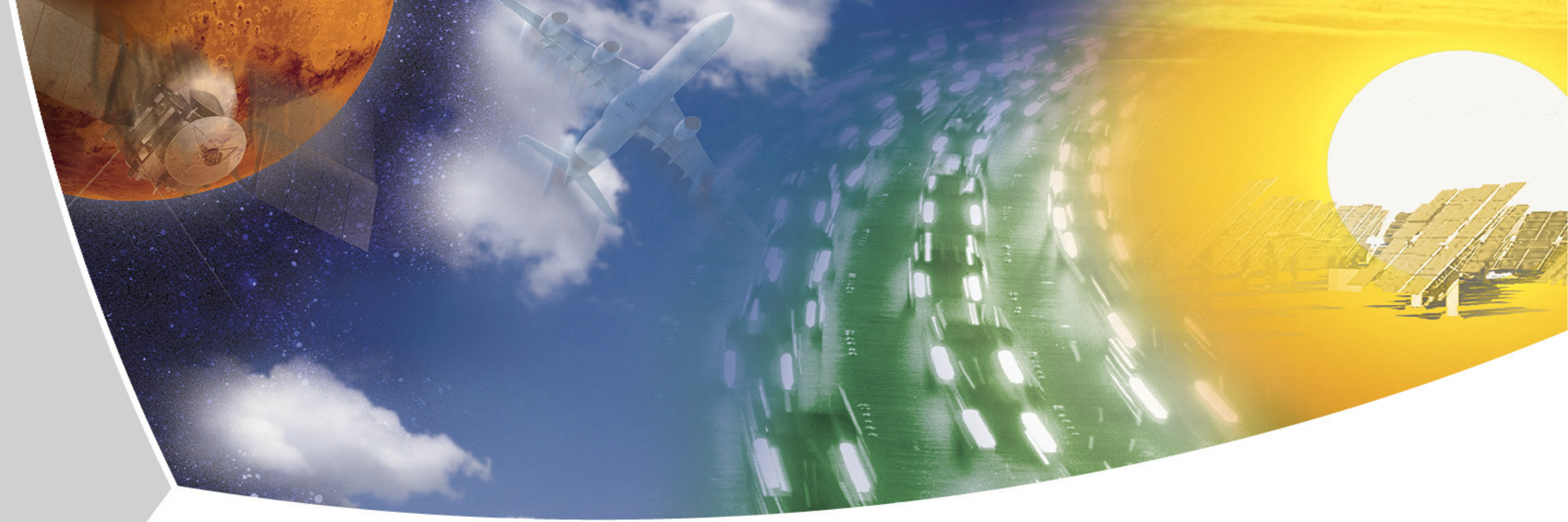

DG methods for aerodynamic flow simulations: Higher order, error estimation and adaptivity

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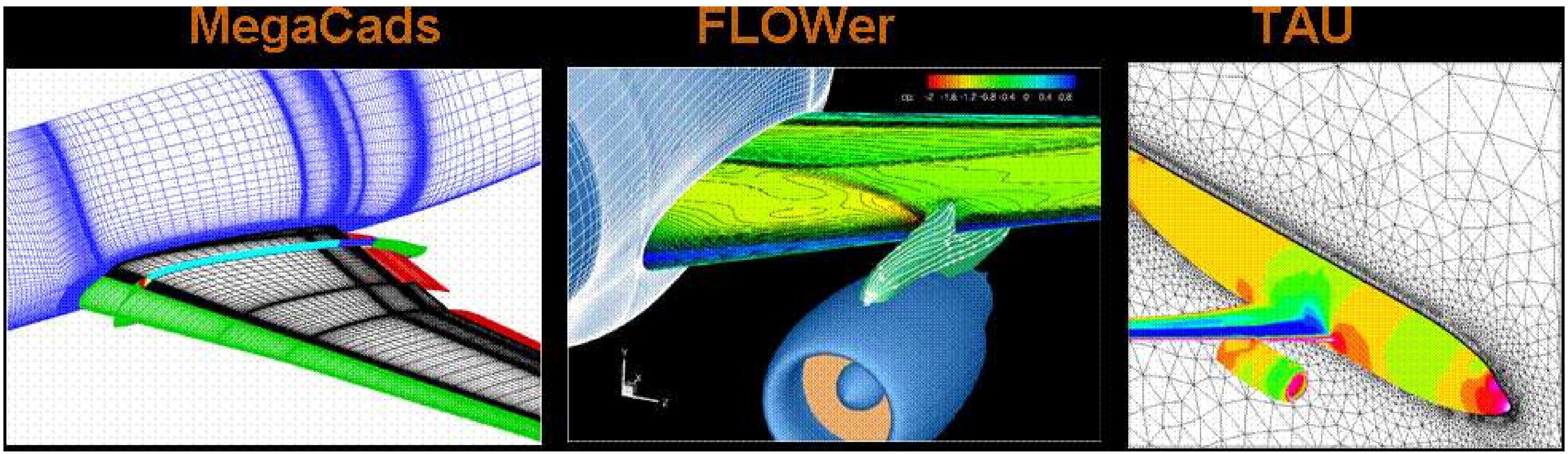
DLR, Institute of Aerodynamics and Flow Technology

- structured grid generator: MegaCads
- structured Finite Volume flow solver: Flower
- unstructured Finite Volume flow solver: TAU



used in industry: Airbus, EADS-M, ...

Motivation

Higher order methods:

- Numerical resolution and tracking of vortices
  - Helicopters: Vortex creation and blade-vortex interaction
  - Transport aircrafts: wake-vortices
- Numerical resolution of viscous boundary layers
- Numerical approximation of aerodynamical forces: lift, drag, moments

Error estimation:

- Reliable prediction of aerodynamical forces

Adaptivity:

- Mesh refinement for better resolution of vortices, boundary layers, etc.
- Goal-oriented mesh refinement for accurate approximation of aerodynamical forces

Overview

- The Discontinuous Galerkin discretization of compr. Euler & Navier-Stokes
- Higher order computational results
- A posteriori error estimation
- Isotropic and anisotropic goal-oriented (adjoint-based) adaptivity

The Discontinuous Galerkin discretization

of the compressible Euler & Navier-Stokes equations

The compr. Euler and Navier-Stokes equations in 2D

The compressible Euler equations:

$$\frac{\partial}{\partial t}\begin{pmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho E \end{pmatrix} + \frac{\partial}{\partial x_1}\begin{pmatrix} \varrho v_1 \\ \varrho v_1^2 + p \\ \varrho v_1 v_2 \\ v_1(\varrho E + p) \end{pmatrix} + \frac{\partial}{\partial x_2}\begin{pmatrix} \varrho v_2 \\ \varrho v_1 v_2 \\ \varrho v_2^2 + p \\ v_2(\varrho E + p) \end{pmatrix} = 0$$

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$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0$$

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The compressible Navier-Stokes equations:

$$\frac{\partial}{\partial t}\mathbf{u} + \nabla \cdot \mathcal{F}^c(\mathbf{u}) - \nabla \cdot \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) = 0$$

The compr. Euler and Navier-Stokes equations in 2D

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$$\mathbf{f}_1^v(\mathbf{u}, \nabla\mathbf{u}) = \begin{pmatrix} 0 \\ \tau_{11} \\ \tau_{21} \\ \tau_{11}v_1 + \tau_{12}v_2 + \kappa T_{x_1} \end{pmatrix}, \quad \mathbf{f}_2^v(\mathbf{u}, \nabla\mathbf{u}) = \begin{pmatrix} 0 \\ \tau_{12} \\ \tau_{22} \\ \tau_{21}v_1 + \tau_{22}v_2 + \kappa T_{x_2} \end{pmatrix}.$$

DG discretization of the compr. Euler equations

The problem:

$$\nabla \cdot \mathcal{F}^c(\mathbf{u}) = 0 \quad \textbf{in } \Omega,$$

with $\mathbf{u} = (\varrho, \varrho v_1, \varrho v_2, \rho E)^T$**.**

The discretization of DG(p): Find $\mathbf{u}_h$ in $\mathbf{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) \equiv \sum_{\kappa \in \mathcal{T}_h} \left\{ -\int_\kappa \mathcal{F}^c(\mathbf{u}_h) : \nabla \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \int_{\partial\kappa \setminus \Gamma} \mathcal{H}(\mathbf{u}_h^+, \mathbf{u}_h^-, \mathbf{n}_\kappa) \, \mathbf{v}_h^+ \, \mathrm{d}s \right\}$$
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$$\mathbf{V}_{h,p} = \left\{ \mathbf{v} \in [L_2(\Omega)]^4 : \mathbf{v}|_\kappa \in [\mathcal{Q}_p(\kappa)]^4 \;\; \forall \kappa \in \mathcal{T}_h \right\},$$

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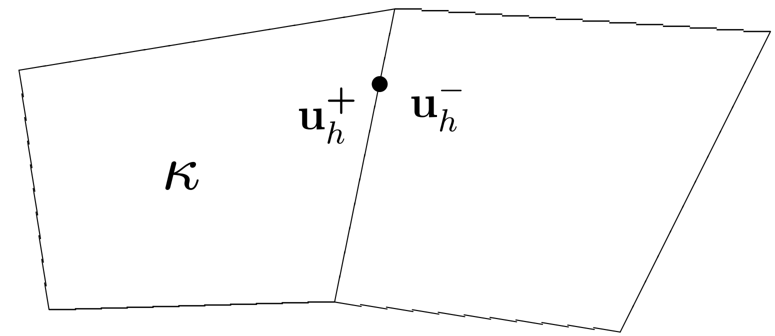

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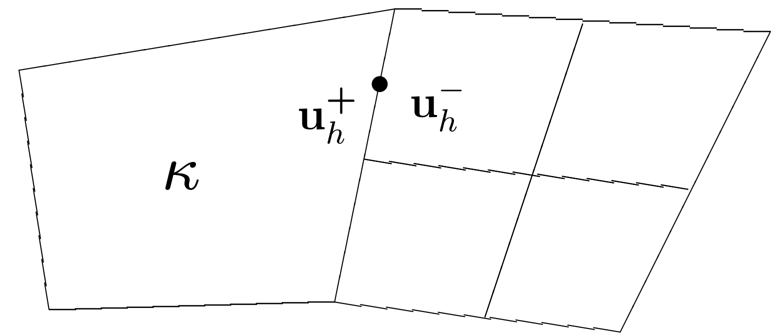

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DG discretization of the compr. Navier-Stokes equations

Find $\mathbf{u}_h$ in $\mathbf{V}_{h,p}$ such that

$$\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) \equiv -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa\setminus\Gamma} \mathcal{H}(\mathbf{u}_h^+,\mathbf{u}_h^-,\mathbf{n}_\kappa)\cdot \mathbf{v}_h^+ \,\mathrm{d}s$$
$$+ \int_\Omega \mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h) : \nabla_h \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_{\Gamma_\mathcal{I}} \{\!\{G(\mathbf{u}_h)\nabla\mathbf{u}_h\}\!\} : \underline{[\![\mathbf{v}_h]\!]} \,\mathrm{d}s$$
$$- \int_{\Gamma_\mathcal{I}} \{\!\{G^\top(\mathbf{u}_h)\nabla\mathbf{v}_h\}\!\} : \underline{[\![\mathbf{u}_h]\!]}) \,\mathrm{d}s + \int_{\Gamma_\mathcal{I}} \delta(\mathbf{u}_h) : \underline{[\![\mathbf{v}_h]\!]} \,\mathrm{d}s + \mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) = 0$$

for all $\mathbf{v}_h$ in $\mathbf{V}_{h,p}$, where the penalization term is given by

$\underline{\delta}(\mathbf{u}_h) = \tilde{C}_{ip}\frac{p^2}{h_e}\underline{[\![\mathbf{u}_h]\!]}$ **for the standard IP scheme,** **used in [Hartmann,Houston2006],**

$\underline{\delta}(\mathbf{u}_h) = C_{ip}\frac{p^2}{h_e}\{\!\{G(\mathbf{u}_h)\}\!\}\underline{[\![\mathbf{u}_h]\!]}$ **for the new IP scheme,** **see [Hartmann,Houston2007],**

$\underline{\delta}(\mathbf{u}_h) = \eta_e\{\!\{\underline{L}_0^e(\mathbf{u}_h)\}\!\}$ **for the BR2 scheme,** **see [Bassi,Rebay et al. 1997],**

and the local lifting operator is defined by: find $\underline{L}_0^e(\mathbf{u}_h) \in \underline{\Sigma}_{h,p}^d$ such that

$$\int_\Omega \underline{L}_0^e(\mathbf{u}_h) : \underline{\tau}\,\mathrm{d}\mathbf{x} = \int_e \underline{[\![\mathbf{u}_h]\!]} : \{\!\{G^\top(\mathbf{u}_h)\underline{\tau}\}\!\}\,\mathrm{d}s \qquad \forall\underline{\tau}\in\underline{\Sigma}_{h,p}^d.$$

DG discretization of the compr. Navier-Stokes equations

Find u_h in $\mathrm{V}_{h,p}$ such that , with

$$\begin{aligned}\mathcal{N}_\Gamma(\mathbf{u}_h,\mathbf{v}_h) \equiv \int_\Gamma \mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+),\mathbf{n})\cdot\mathbf{v}_h^+\,\mathrm{d}s + \int_\Gamma \delta_\Gamma(\mathbf{u}_h^+)\cdot\mathbf{v}_h^+\,\mathrm{d}s,\\ - \int_\Gamma G_\Gamma(\mathbf{u}_h)\nabla\mathbf{u}_h : \underline{[\![\mathbf{v}_h]\!]}\,\mathrm{d}s\\ - \int_\Gamma G_\Gamma^\top(\mathbf{u}_h)\nabla\mathbf{v}_h : \Big(\mathbf{u}_h^+ - \mathbf{u}_\Gamma(\mathbf{u}_h^+)\Big)\otimes\mathbf{n}\,\mathrm{d}s,\end{aligned}$$

where the boundary function $\mathrm{u}_\Gamma(\mathrm{u}_h)$ realizes the following boundary conditions:

- **supersonic in- or outflow**
- **subsonic in- or outflow**
- **noslip wall boundary condition (adiabatic or isothermal)**

Symmetry boundary conditions according to the discretization on interior faces.

Optimal order DG discretizations: adjoint consistency

For adjoint consistency require, see [Lu,Darmofal2006],

$$\mathcal{H}_\Gamma(\mathbf{u}_h^+, \mathbf{u}_\Gamma(\mathbf{u}_h^+), \mathbf{n}) = \mathbf{n} \cdot \mathcal{F}_\Gamma^c(\mathbf{u}_h^+) = \mathbf{n} \cdot \mathcal{F}^c(\mathbf{u}_\Gamma(\mathbf{u}_h^+)), \quad \textbf{on } \Gamma_W.$$

and the aerodynamical force coefficients must be evaluated as, see [Hartmann2007],

$$\tilde{J}(\mathbf{u}_h) = J(\mathbf{u}_\Gamma(\mathbf{u}_h)) + \int_\Gamma \underline{\delta}_\Gamma(\mathbf{u}_h) : \mathbf{z}_\Gamma \otimes \mathbf{n} \, \mathrm{d}s,$$

where $J(\mathbf{u})$ represents the lift or drag coefficient:

$$J(\mathbf{u}) = \int_{\Gamma_W} (p\, \mathbf{n} - \underline{\tau}\, \mathbf{n}) \cdot \psi_{\Gamma_W} \, \mathrm{d}s.$$

Modification of a specific $J(u)$ was originally given for Poisson's equation in

[Harriman,Gavaghan,Süli2004].

Higher order computational results

Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ flow around the NACA0012 airfoil

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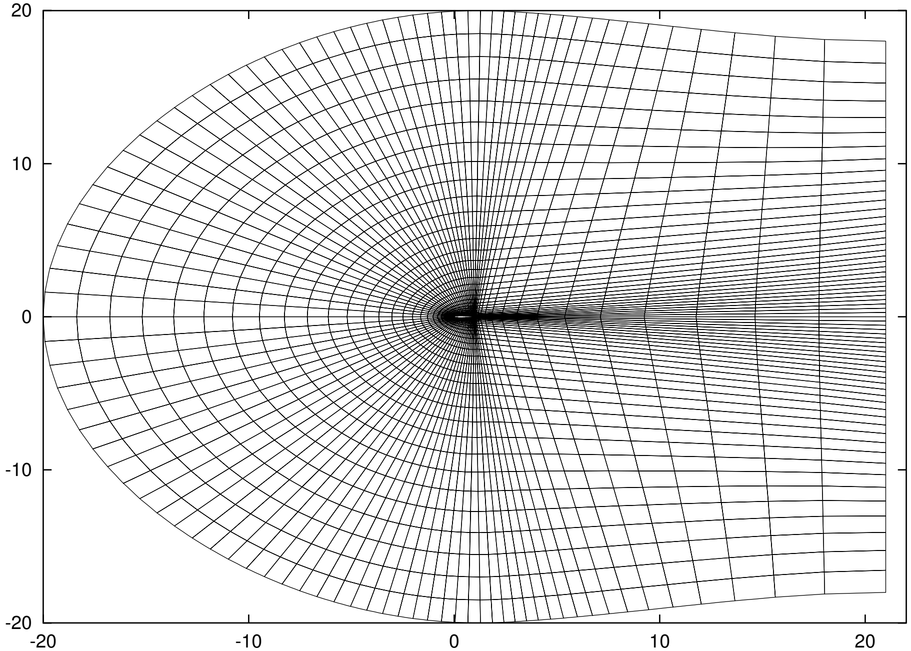
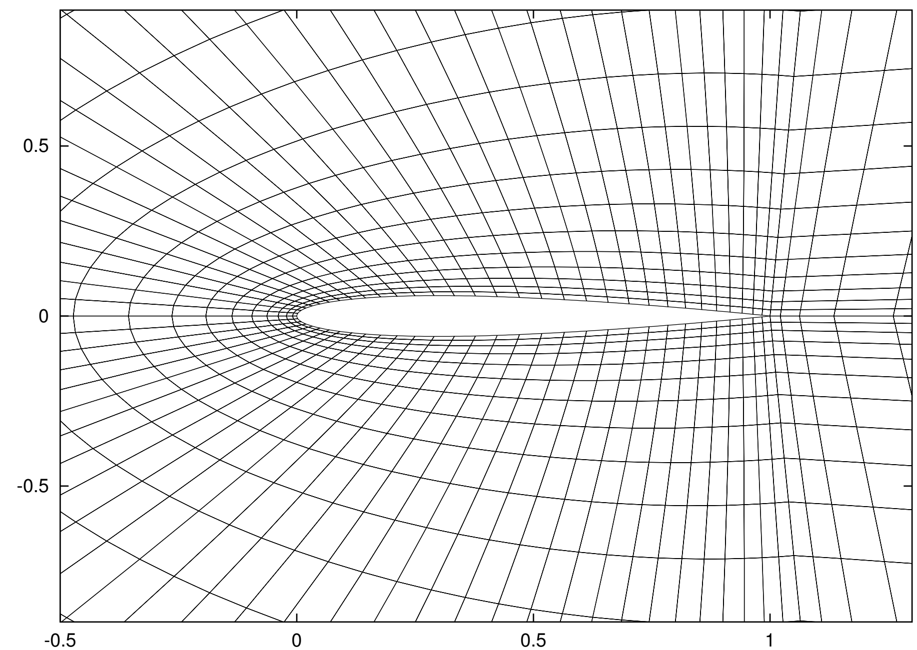
Grid of 3072 cells:



Zoom of this grid:



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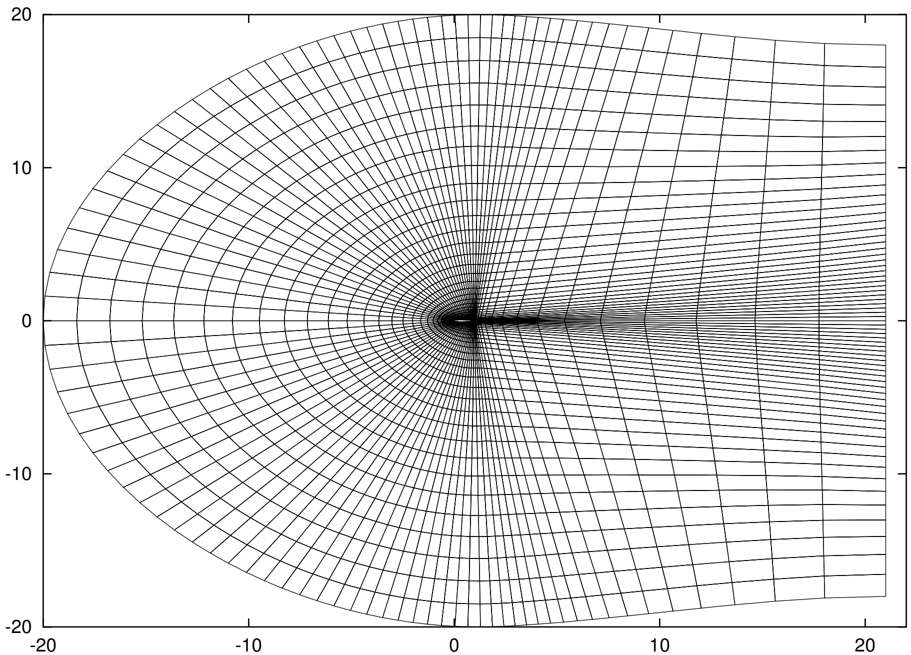

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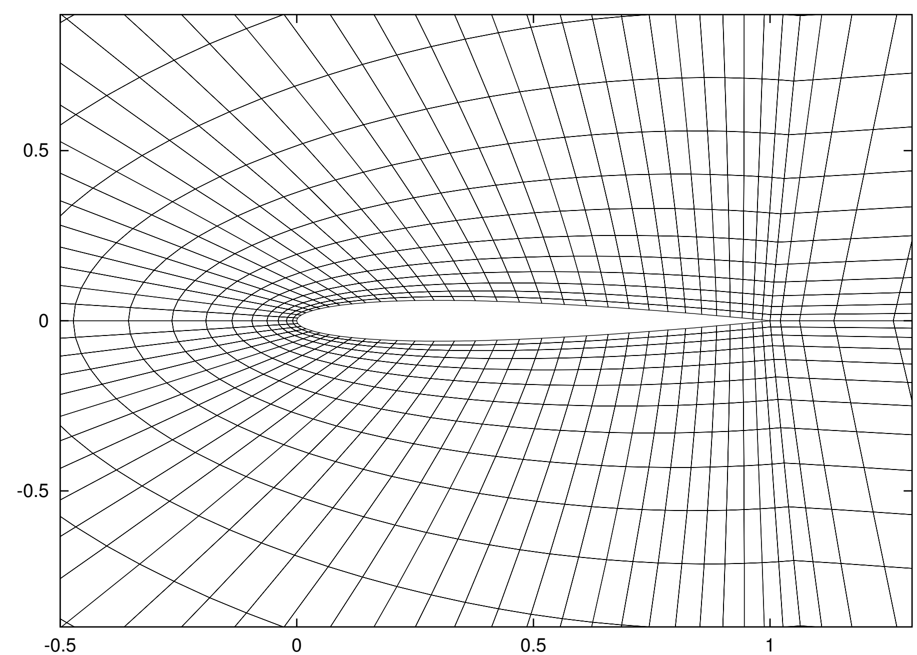

Computation using DG(p), $p = 1, 2, 3$ on sequence of globally refined grids of 3072, 12288, 49152 and 196608 cells.

Higher order computations for laminar test case

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Convergence of cdp and cdf under global refinement, see [Hartmann,Houston2006]

cdp (pressure induced drag)

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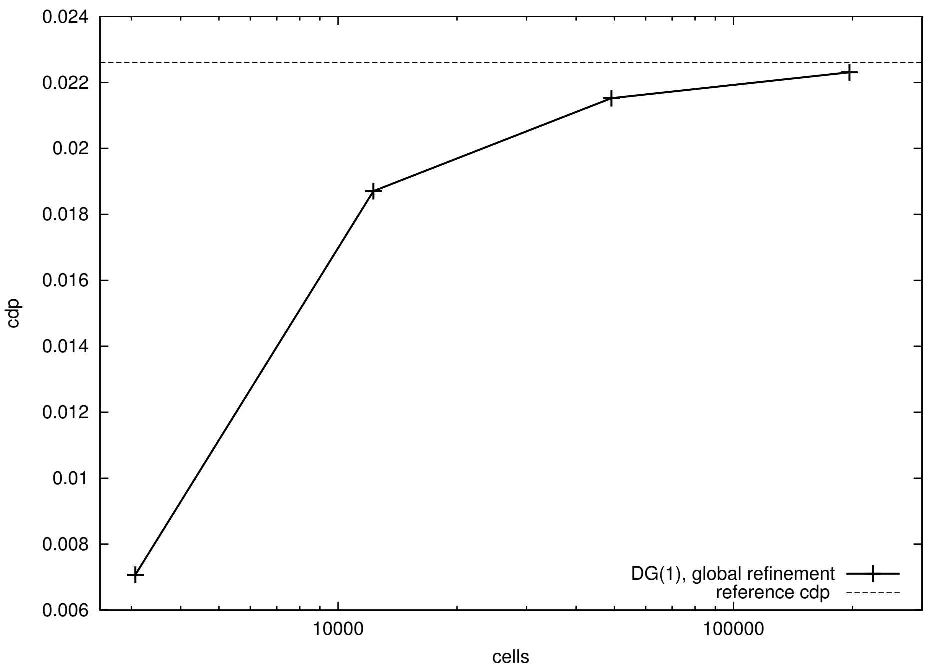

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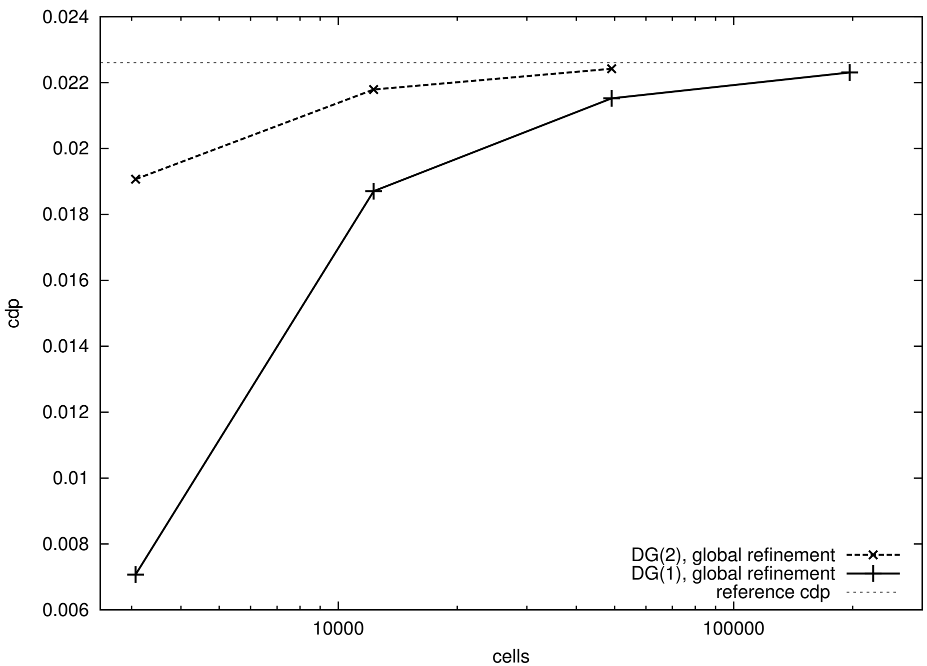

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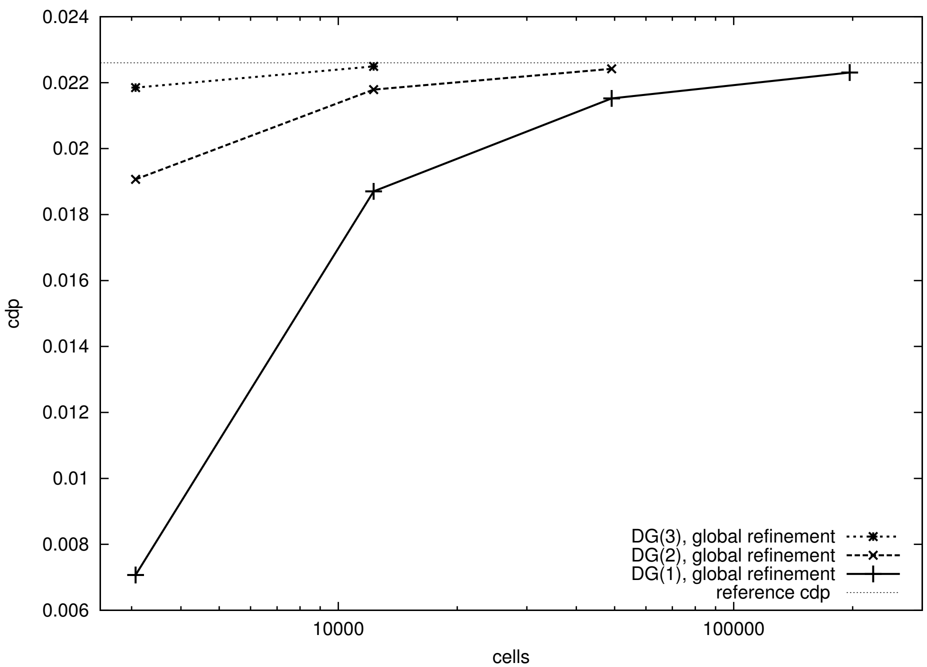

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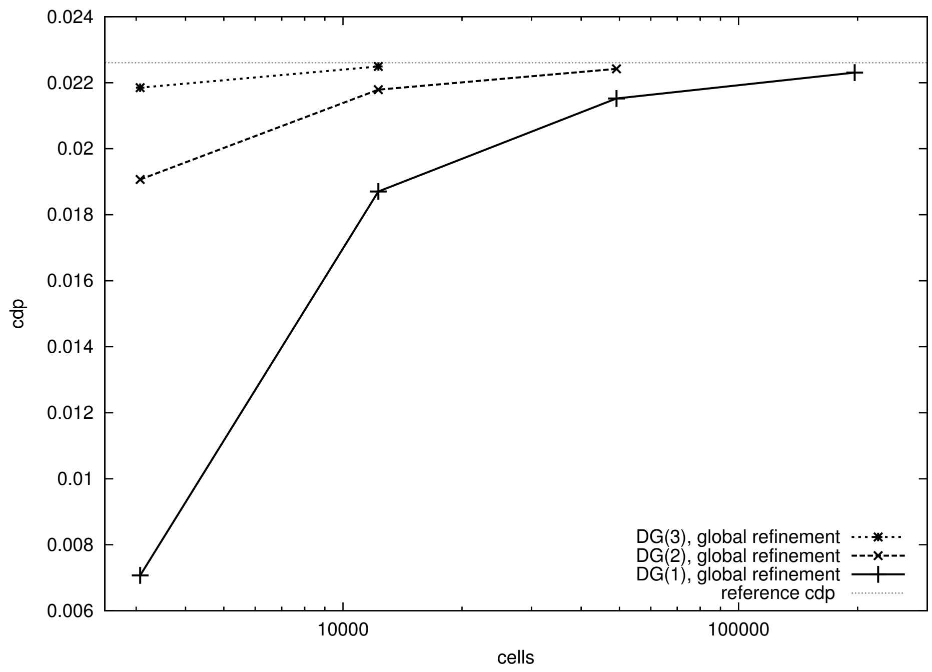
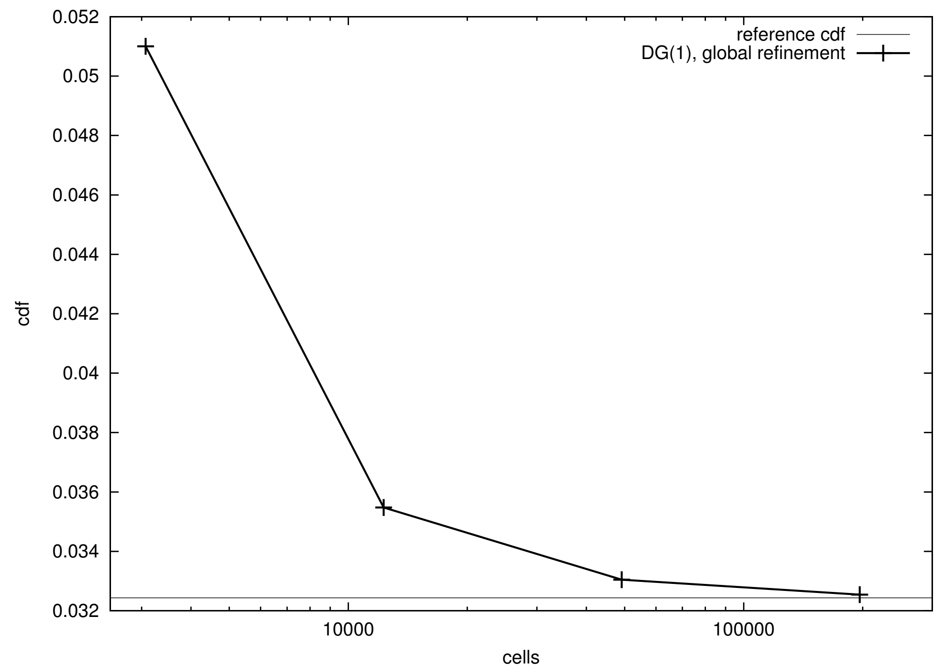


cdf (viscous drag)



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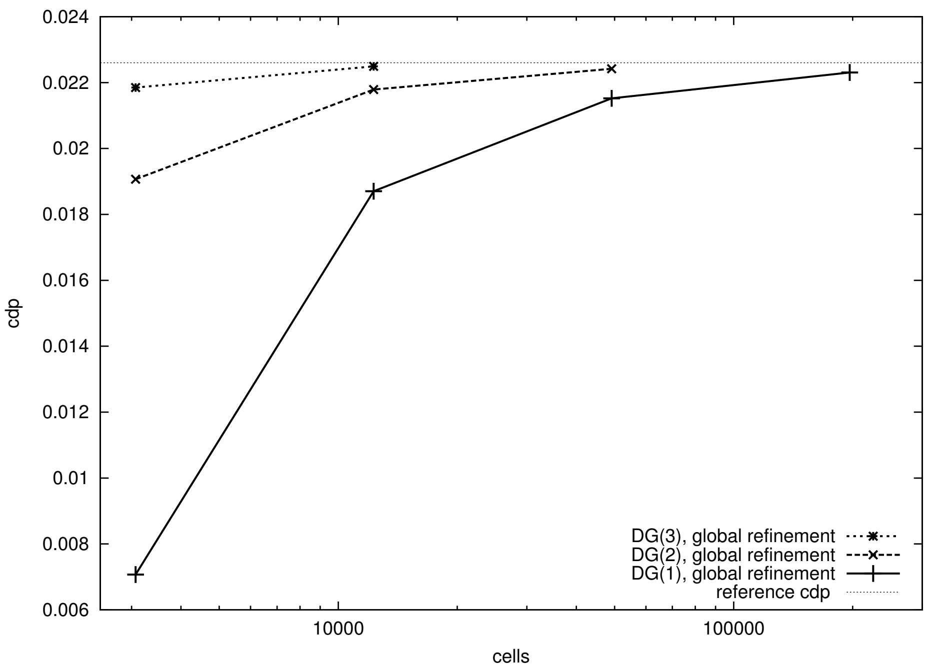

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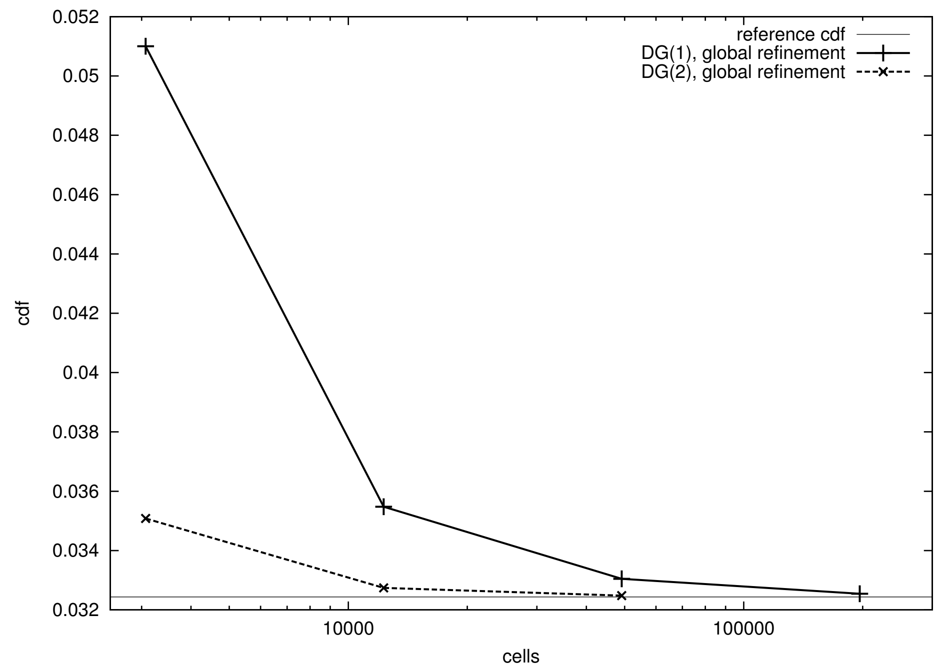

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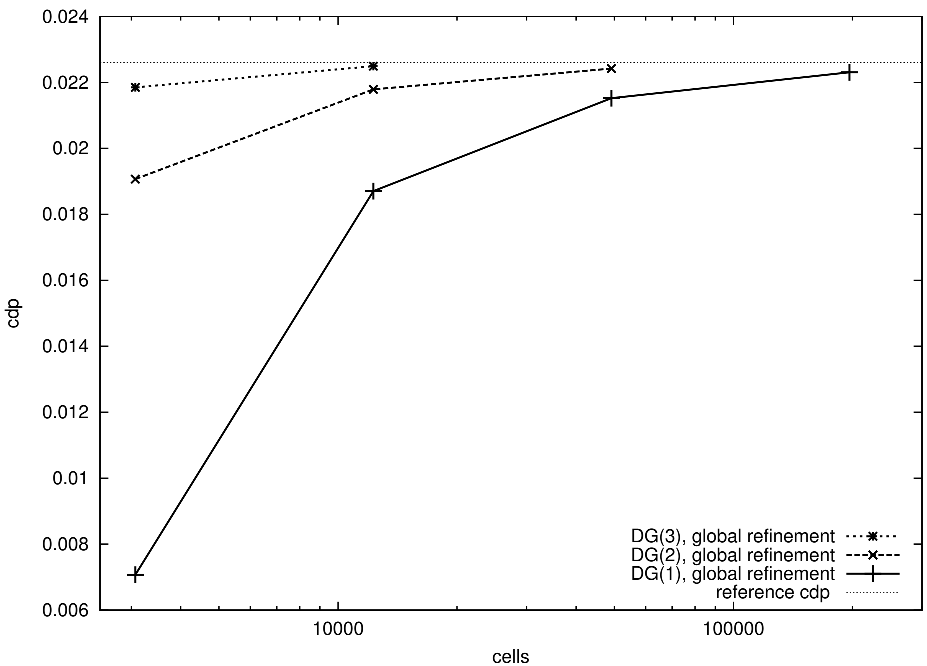

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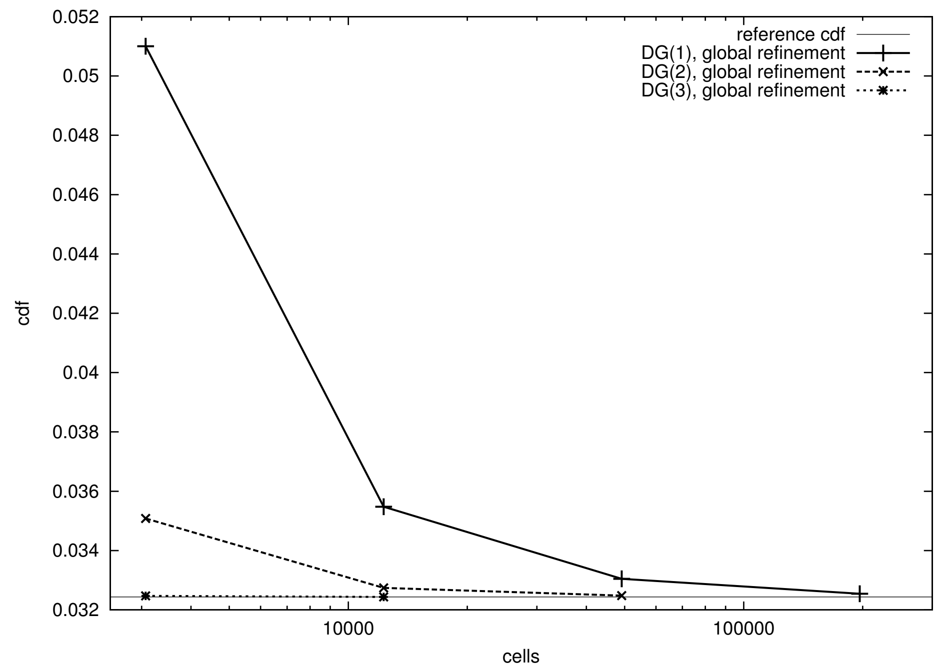
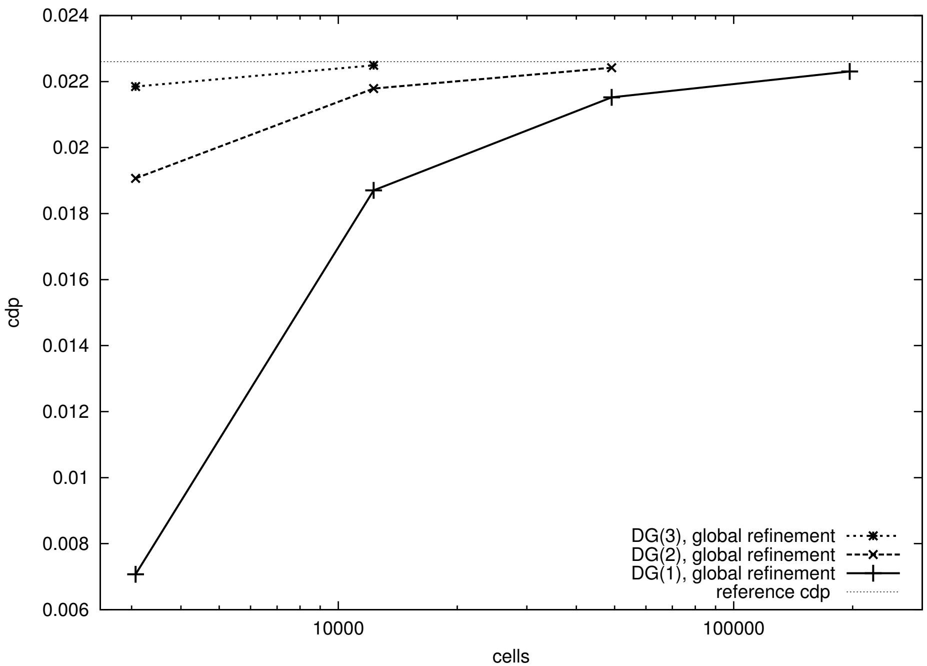


Convergence of cdp

cdp



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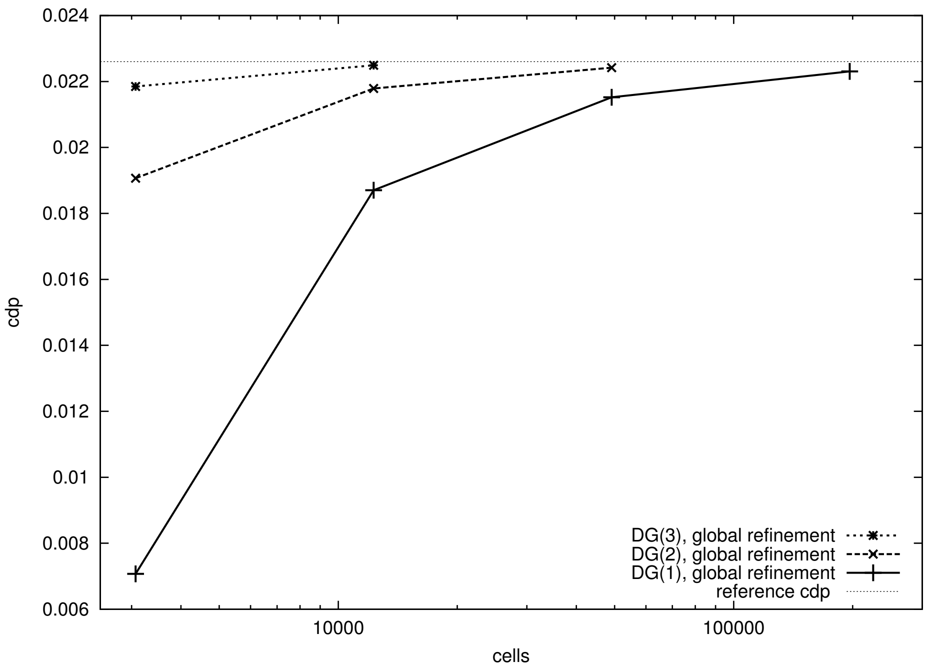
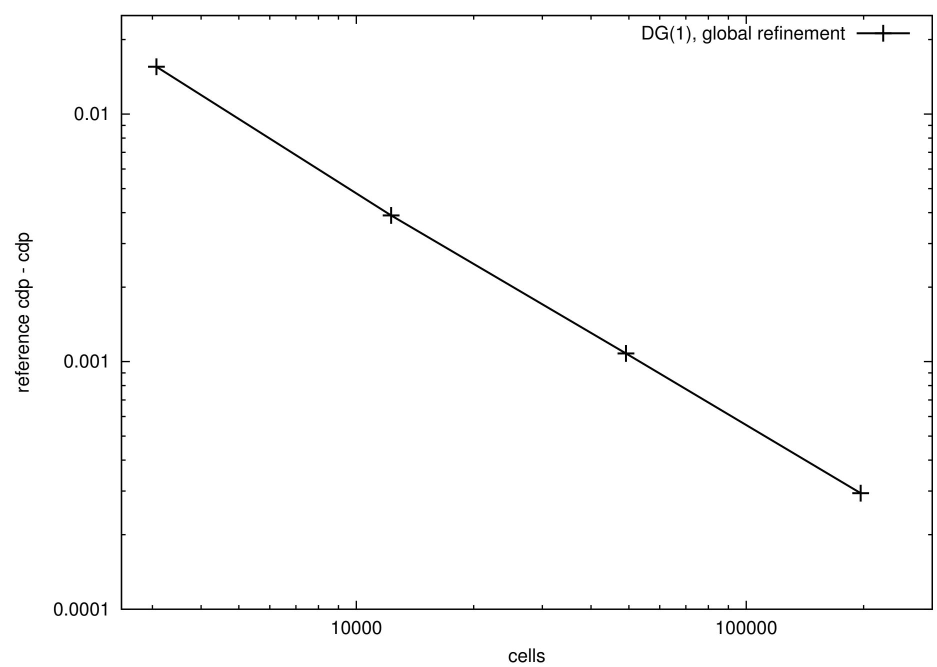


reference cdp - cdp



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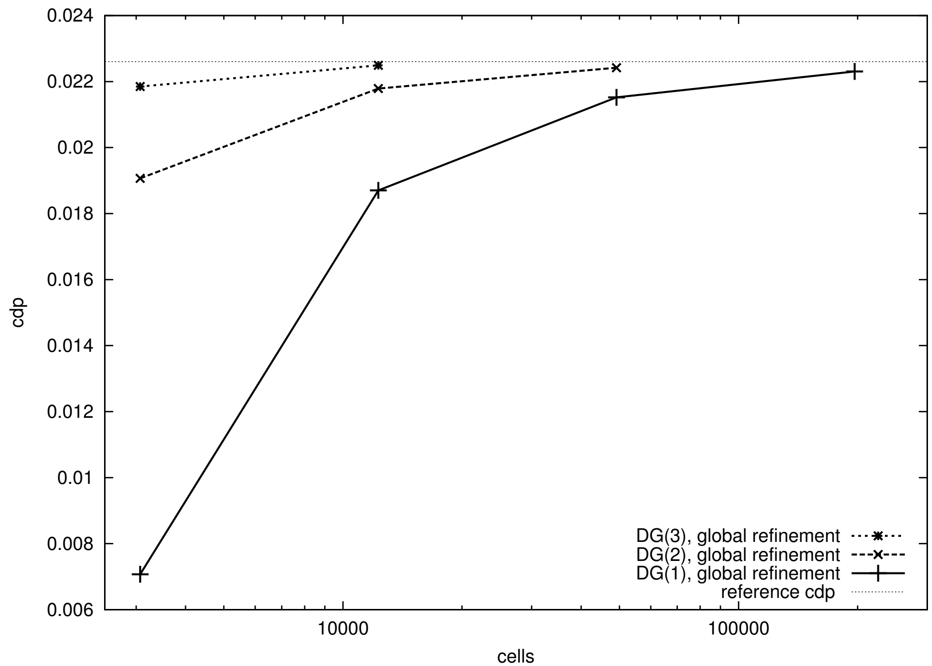

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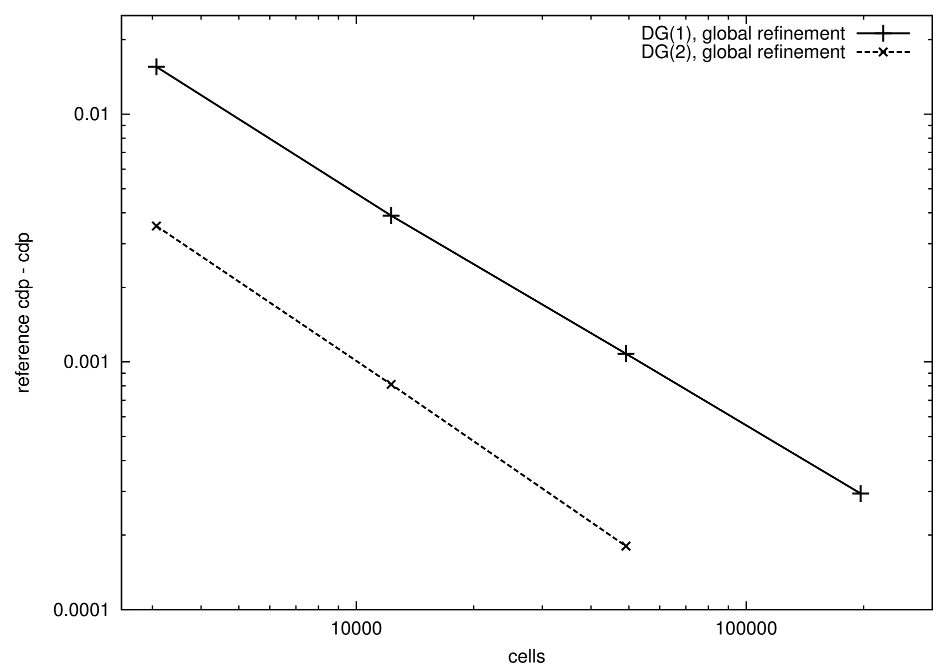

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DLR für Luft- und Raumfahrt e.V.
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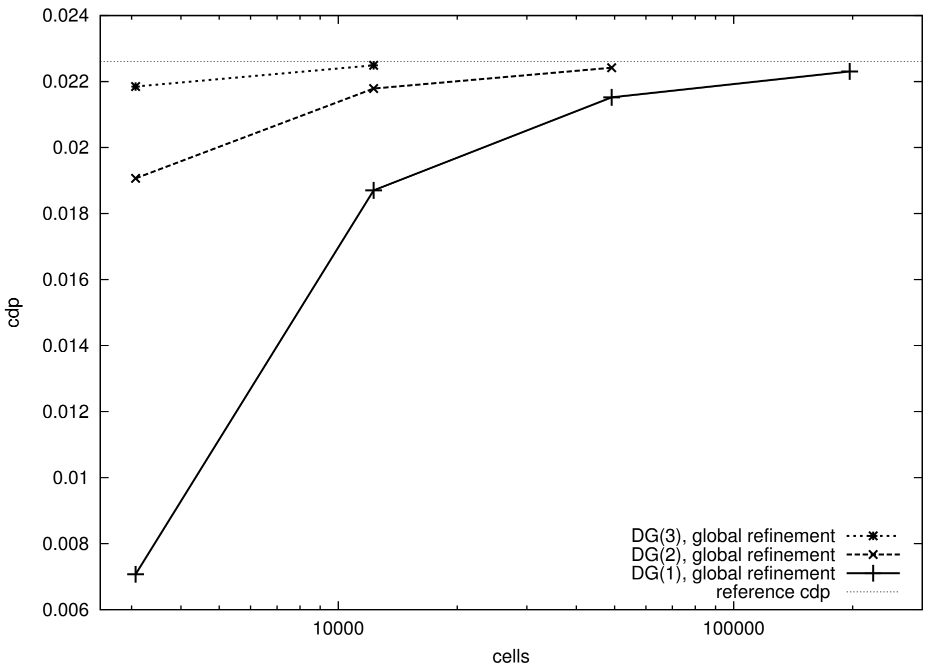

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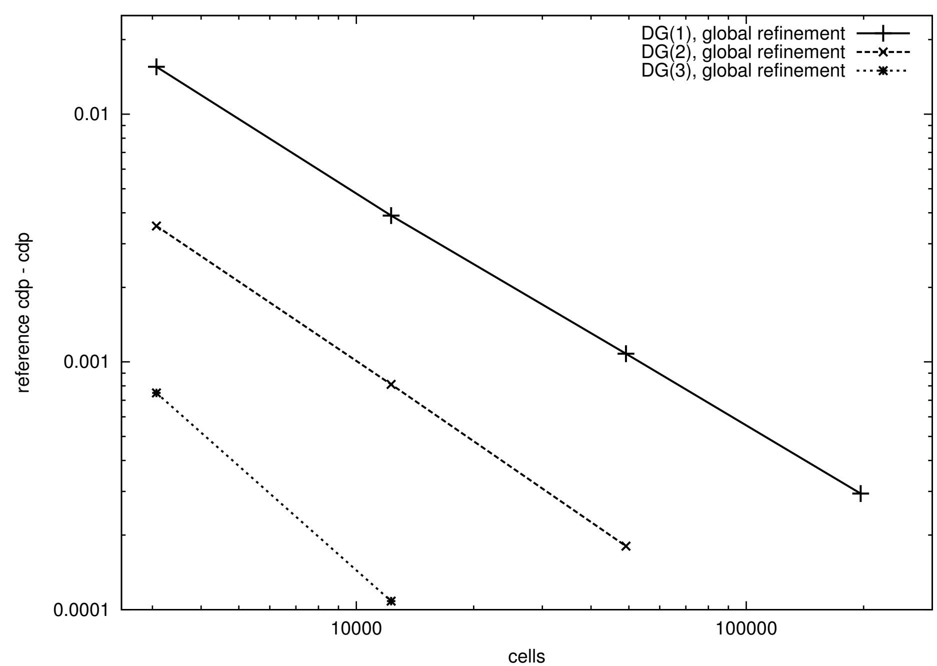

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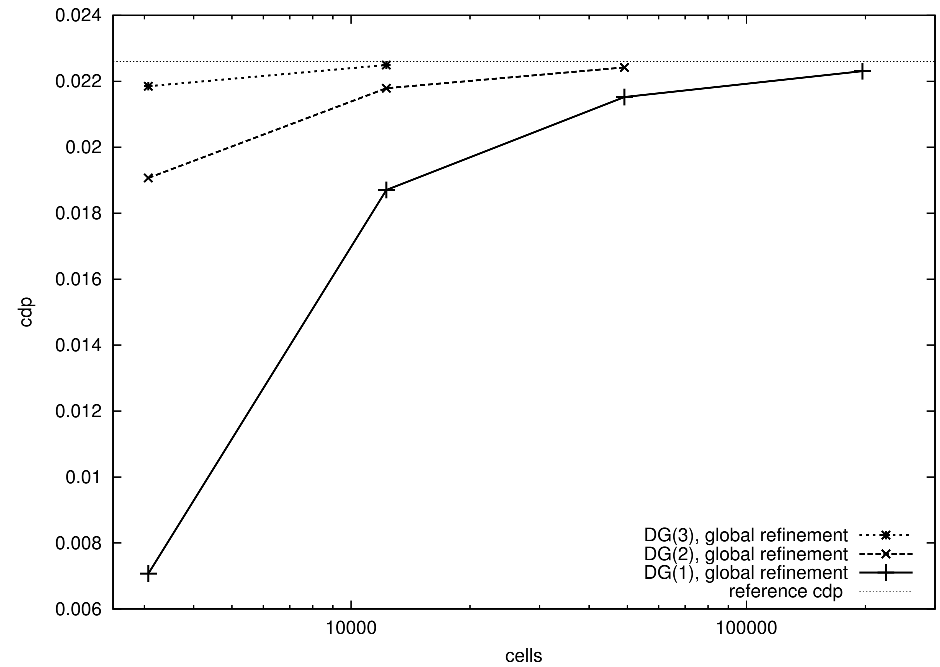

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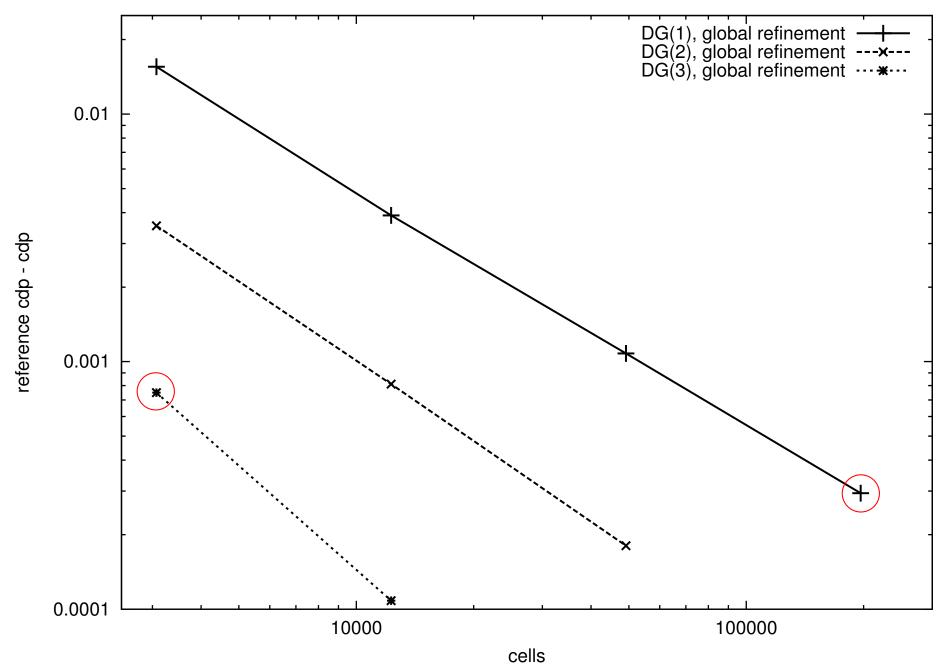

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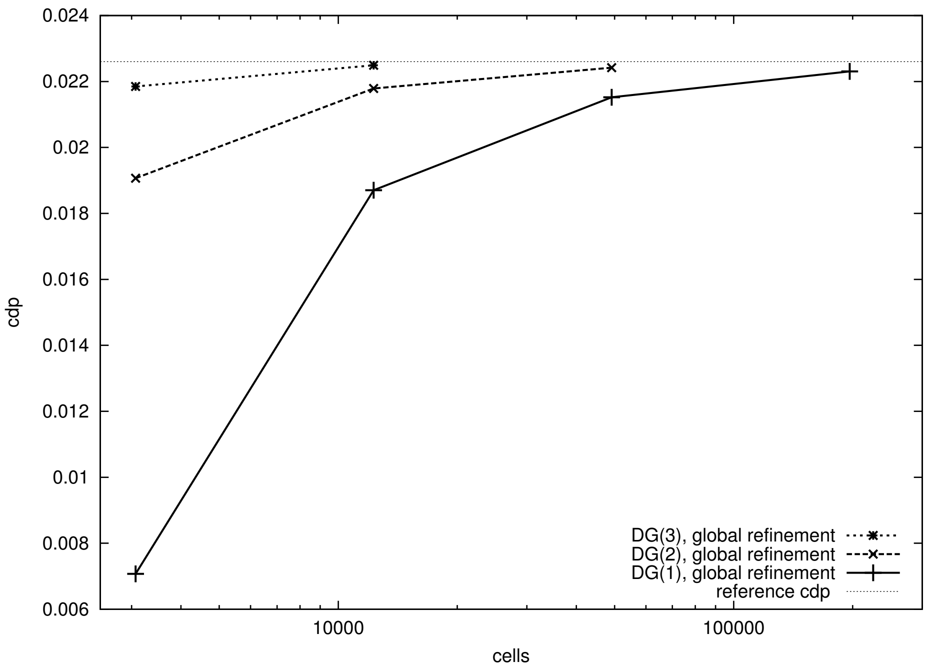

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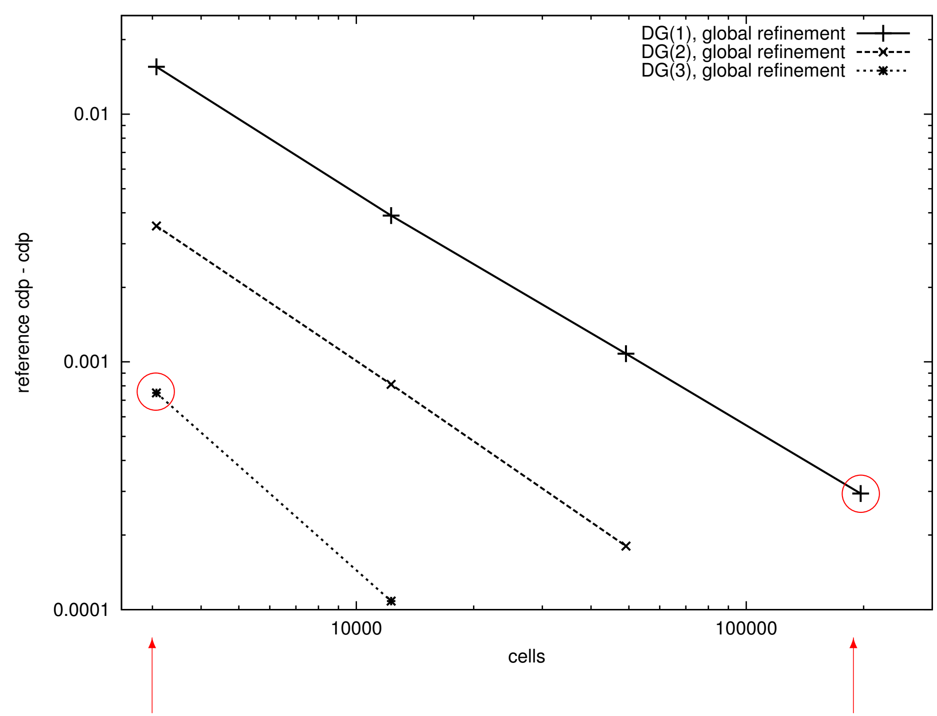

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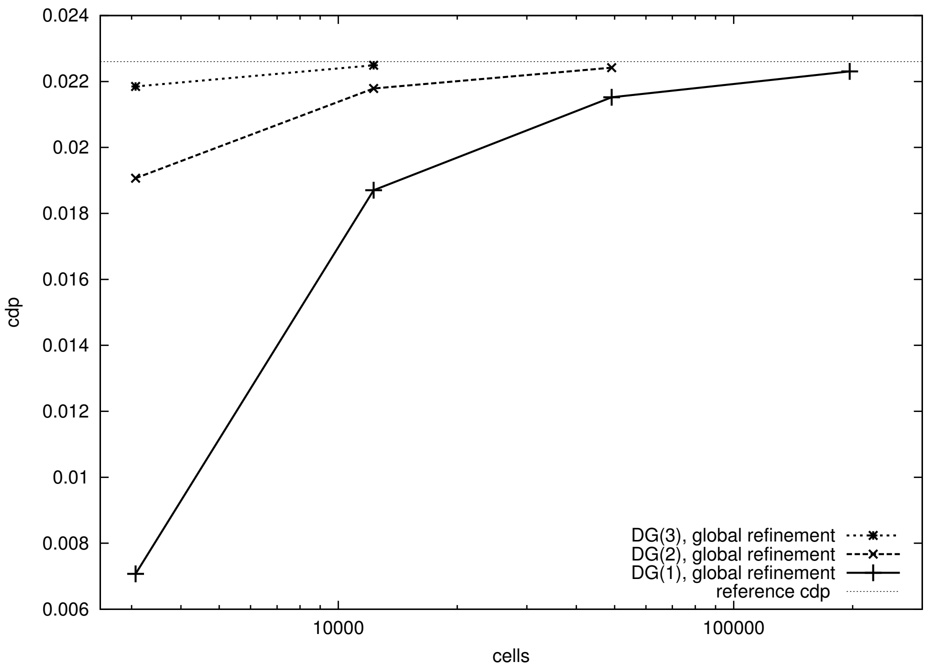

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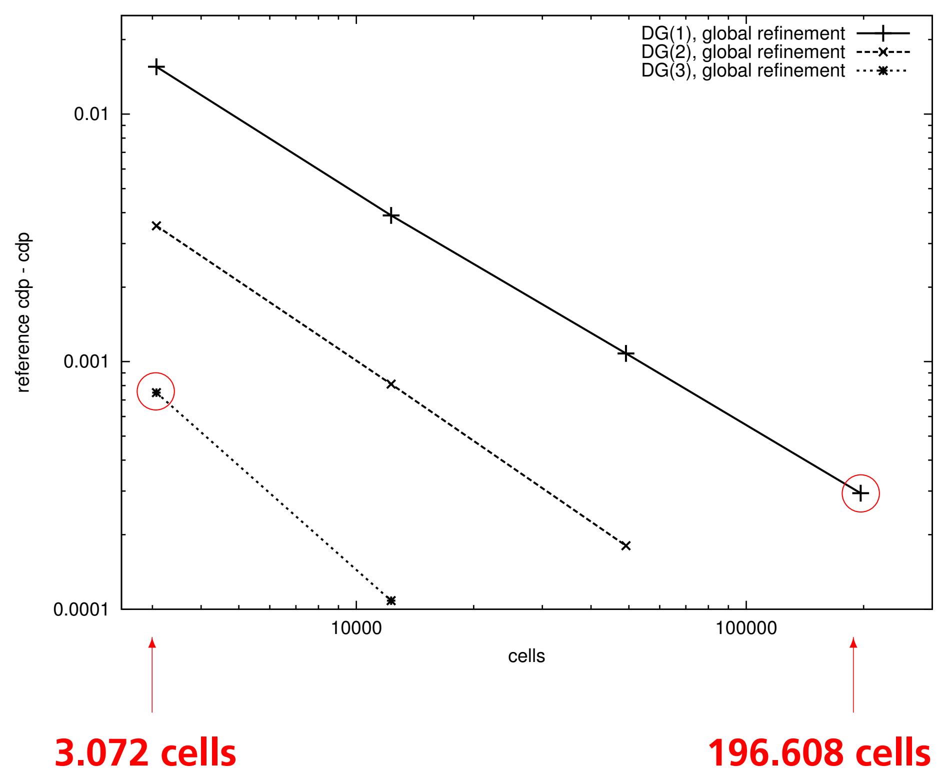

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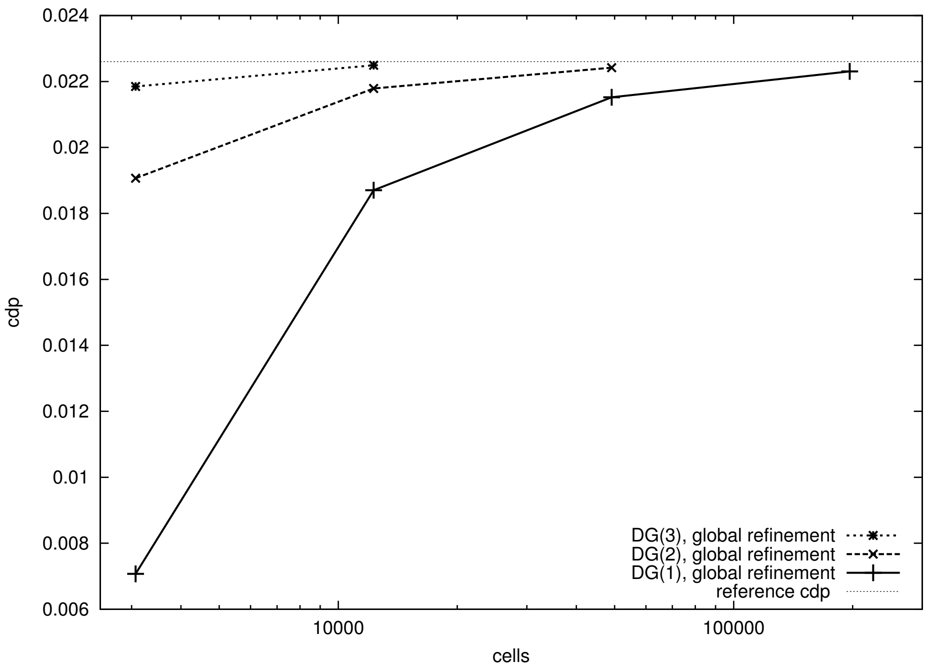
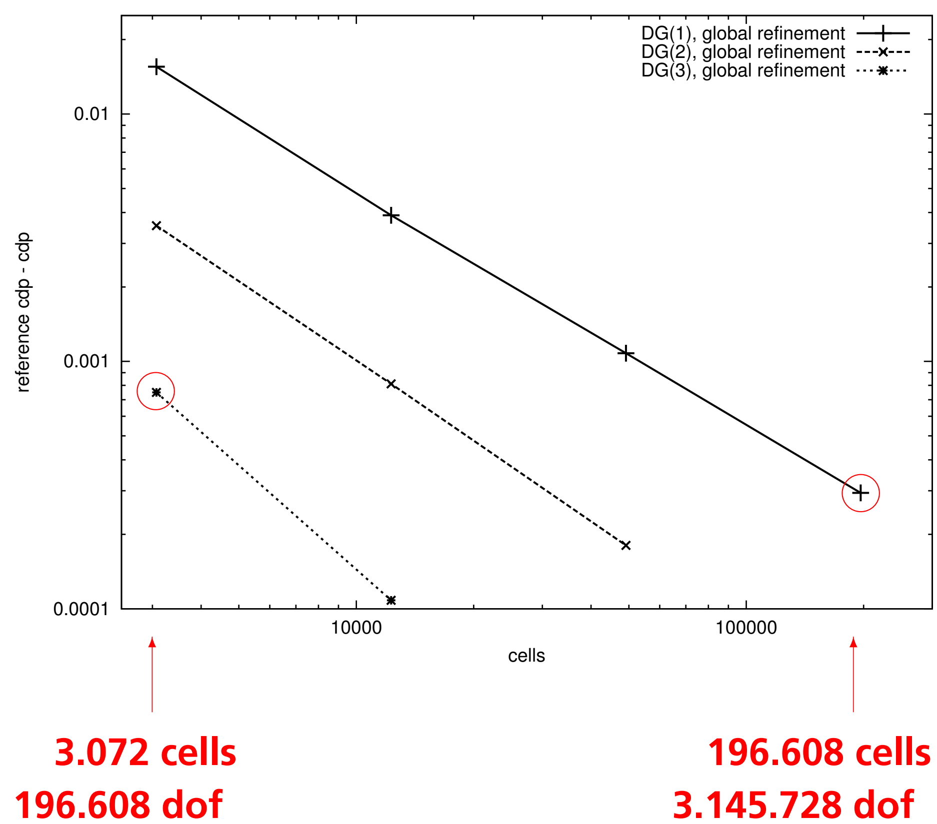
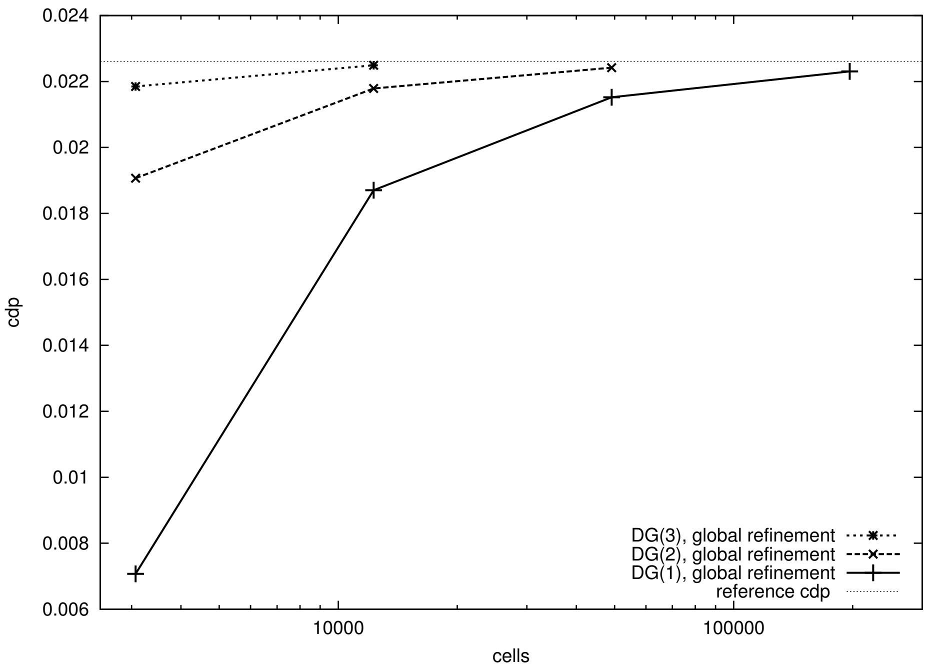


reference cdp - cdp

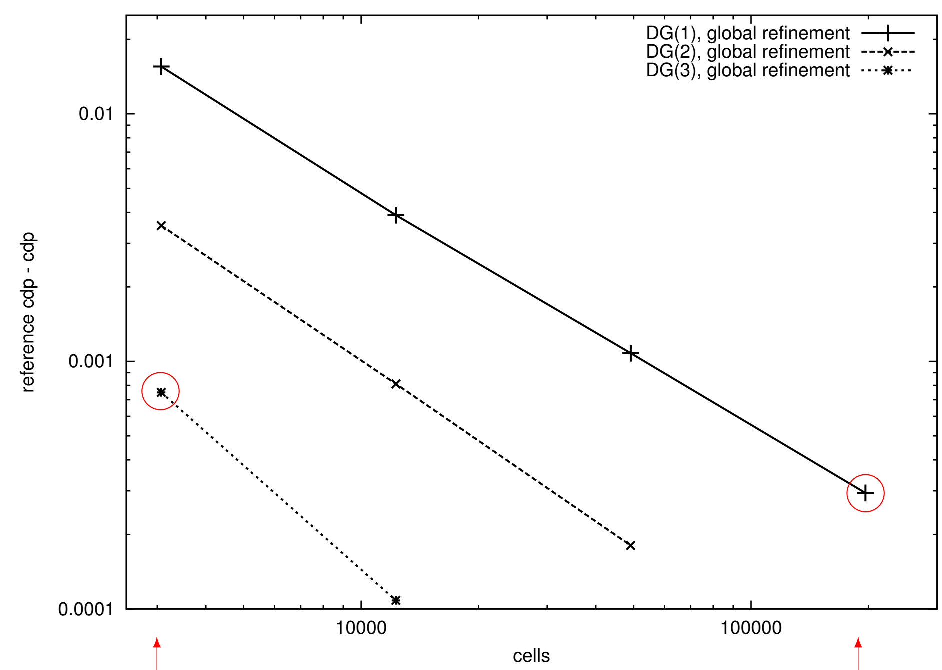


Convergence of cdp

cdp



reference cdp - cdp



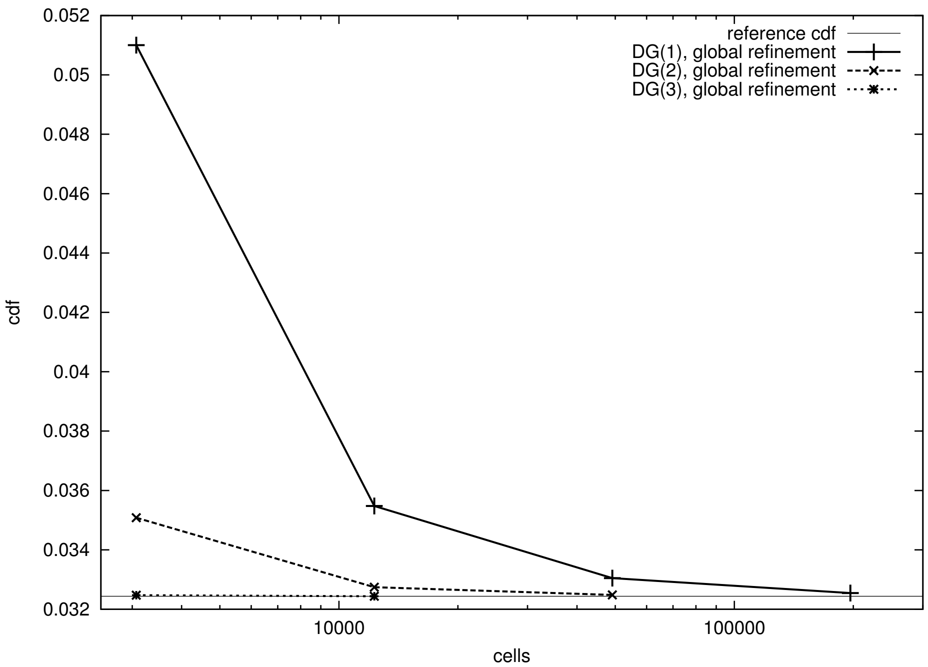
3.072 cells
196.608 dof
8 min

196.608 cells
3.145.728 dof
136 min

Deutsches Zentrum für Luft- und Raumfahrt e.V. in der Helmholtz-Gemeinschaft

Convergence of cdf

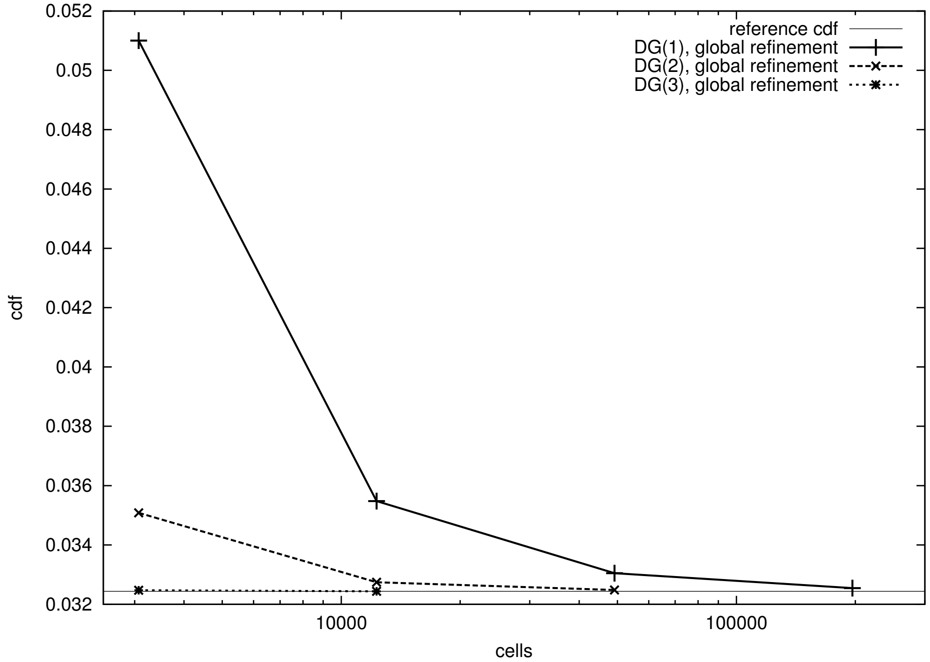
cdf



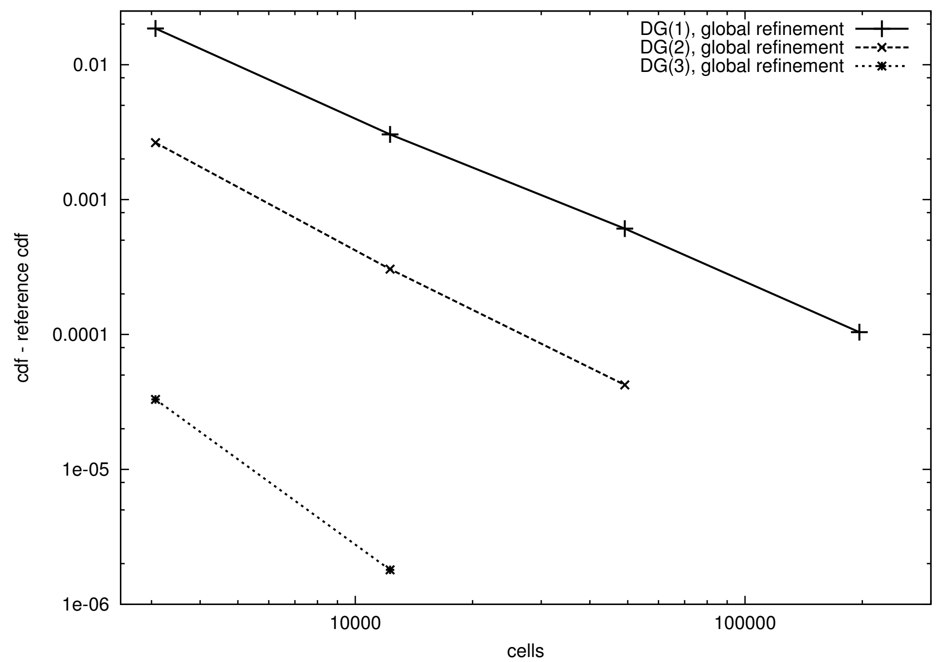
Deutsches Zentrum für Luft- und Raumfahrt e.V. in der Helmholtz-Gemeinschaft

Convergence of cdf

cdf

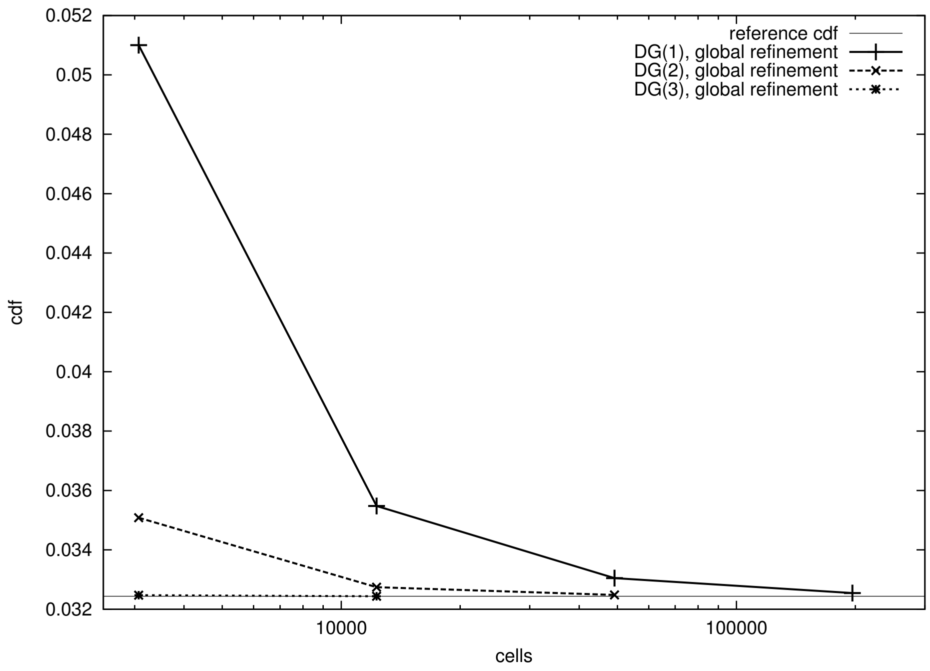


cdf - reference cdf

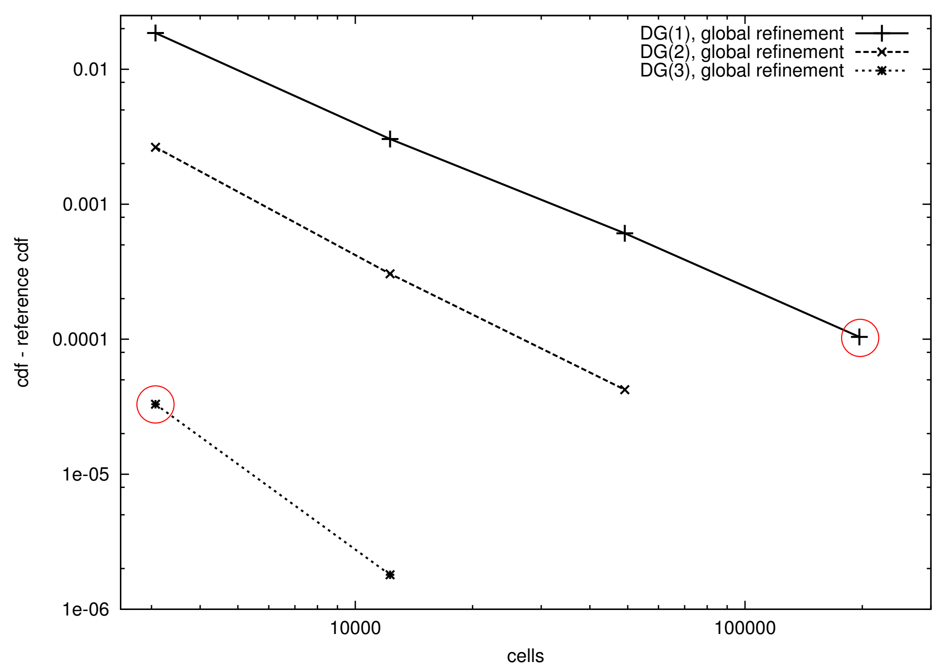


Convergence of cdf

cdf

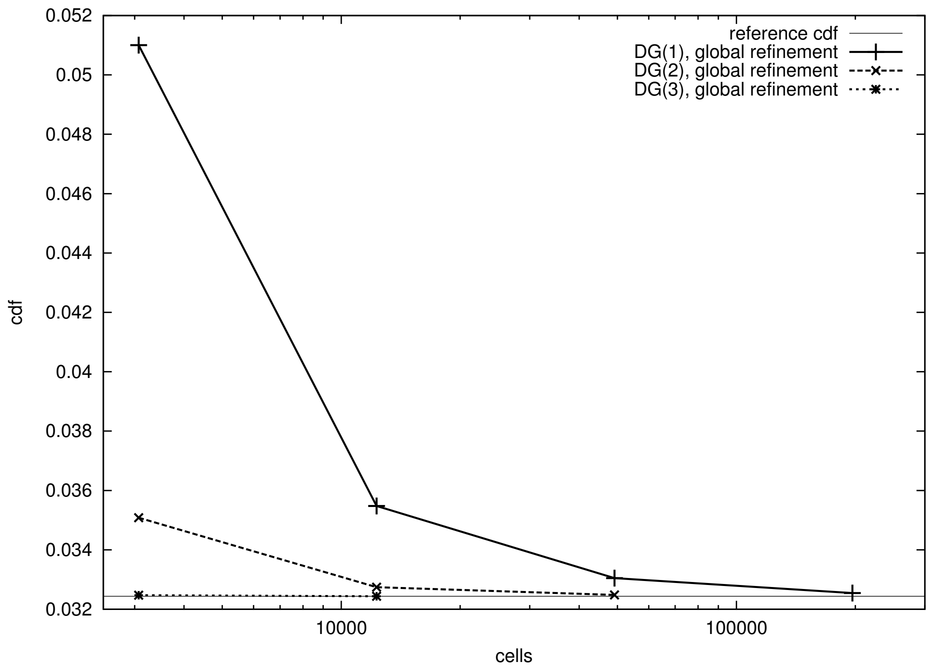


cdf - reference cdf

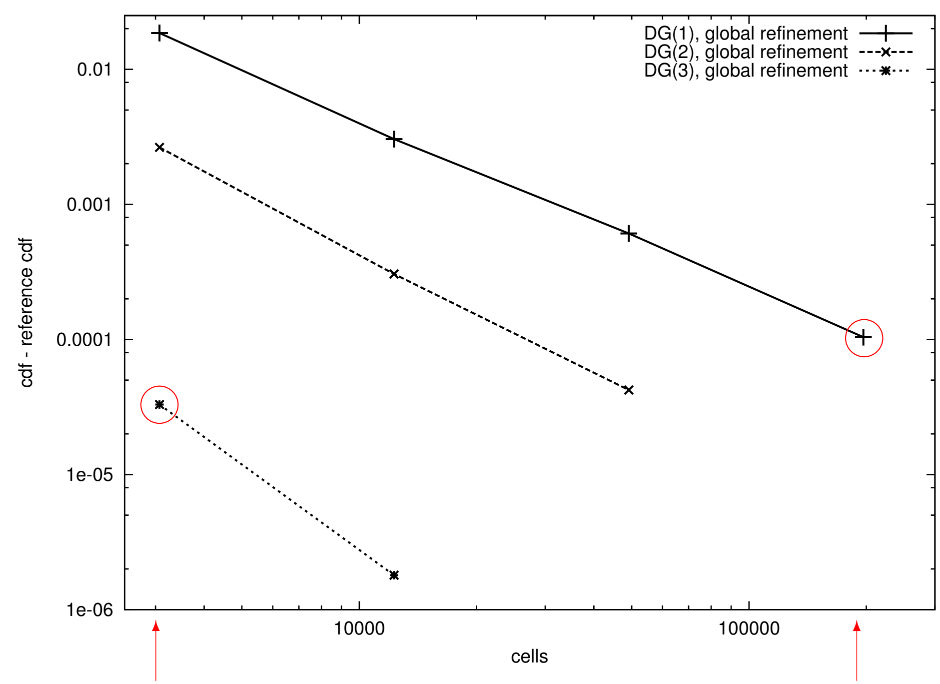


Convergence of cdf

cdf

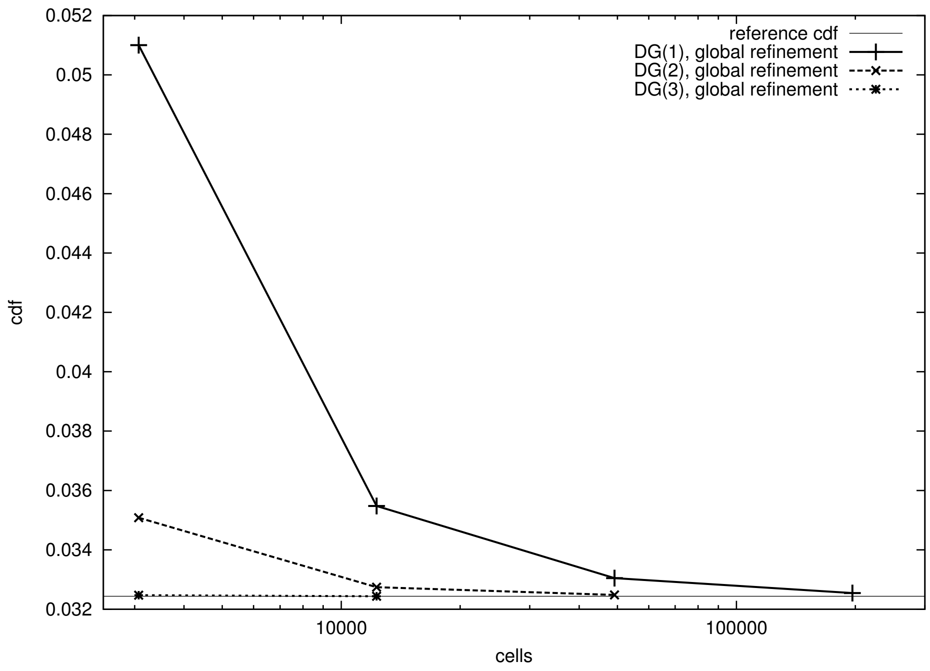


cdf - reference cdf

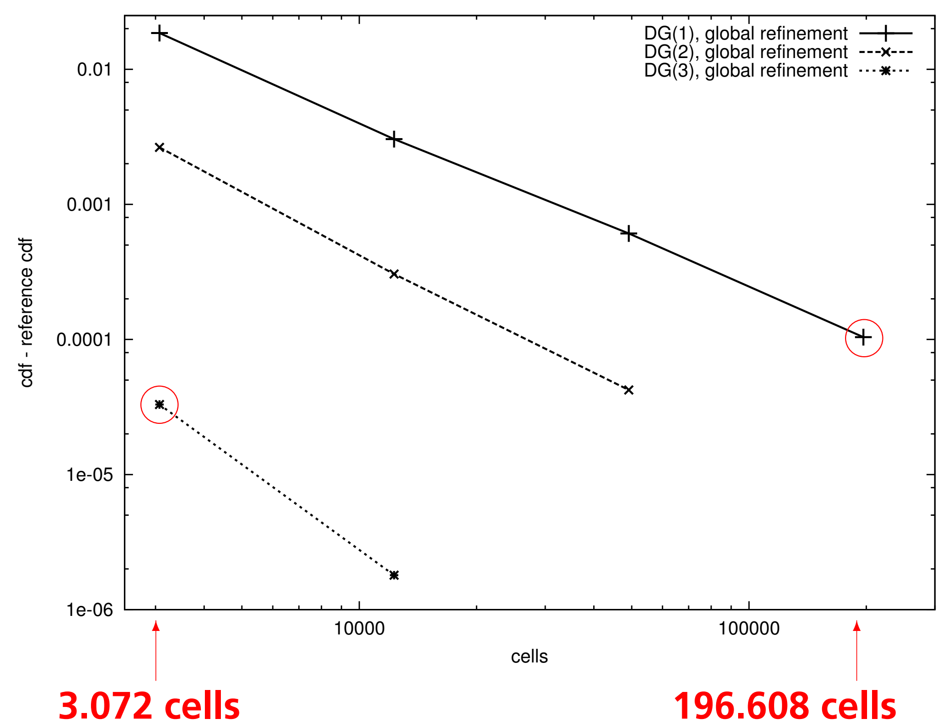


Convergence of cdf

cdf

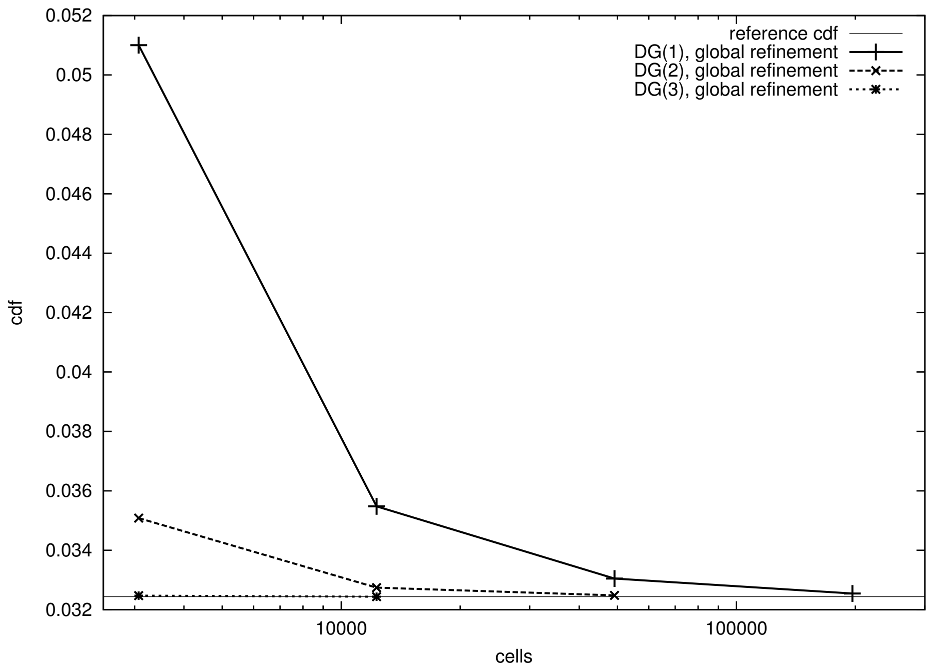


cdf - reference cdf

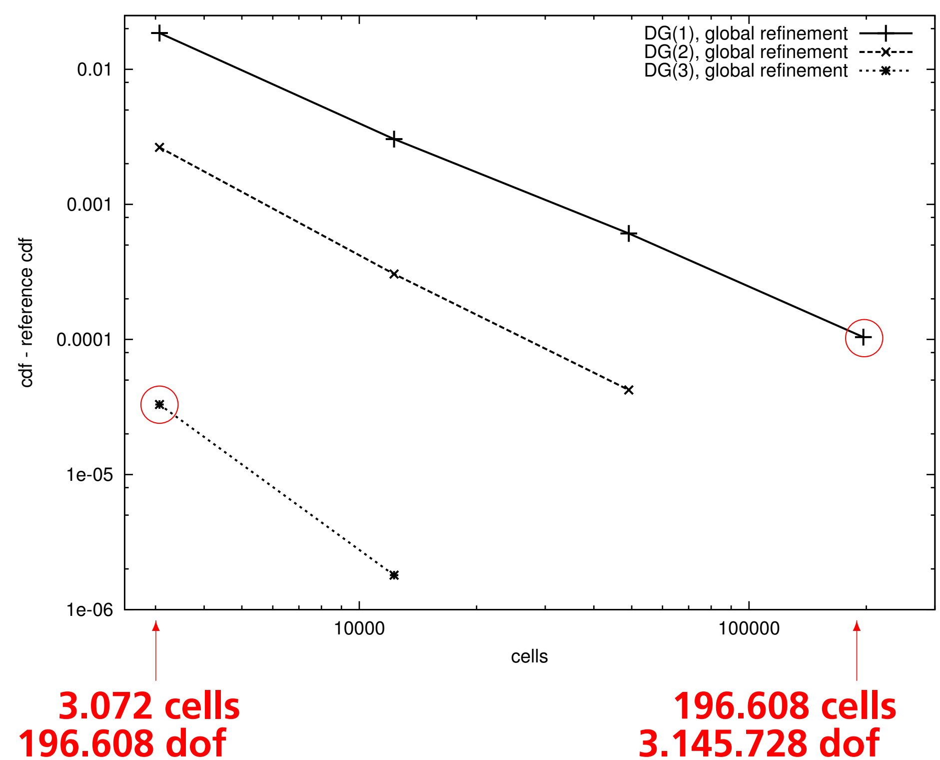


Convergence of cdf

cdf

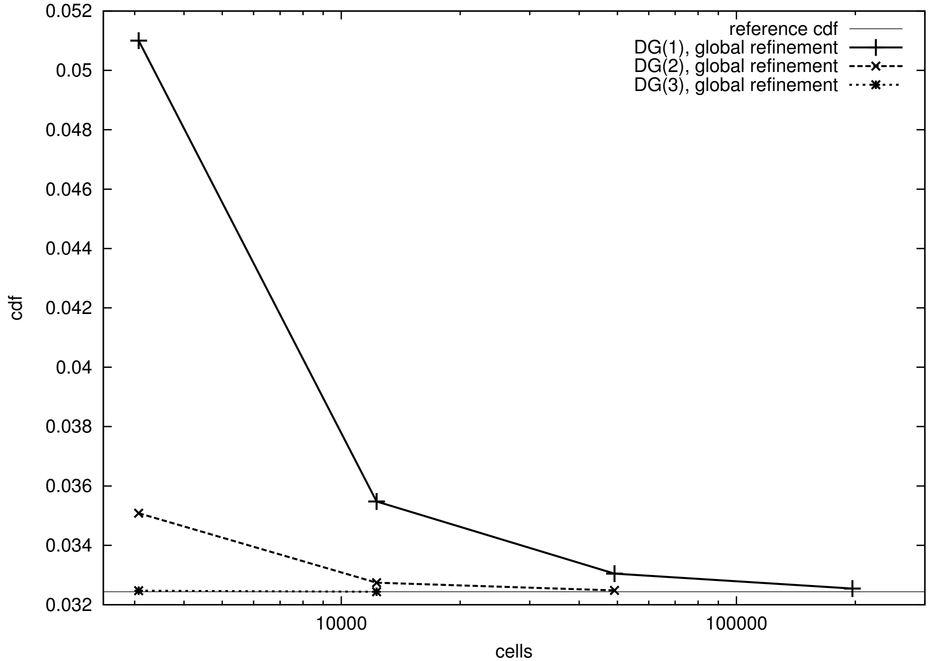


cdf - reference cdf

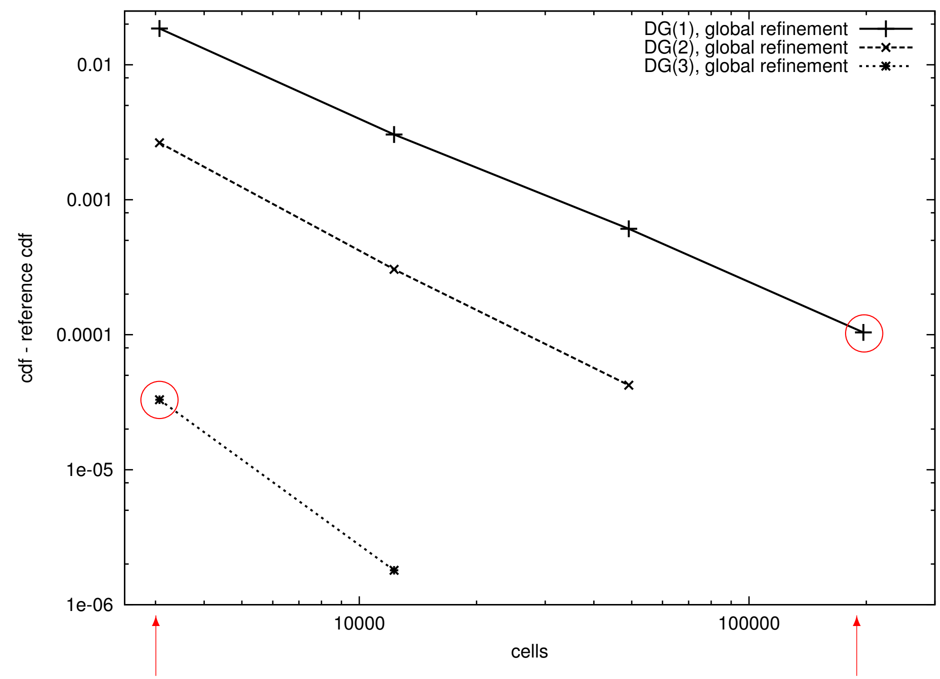


Convergence of cdf

cdf



cdf - reference cdf

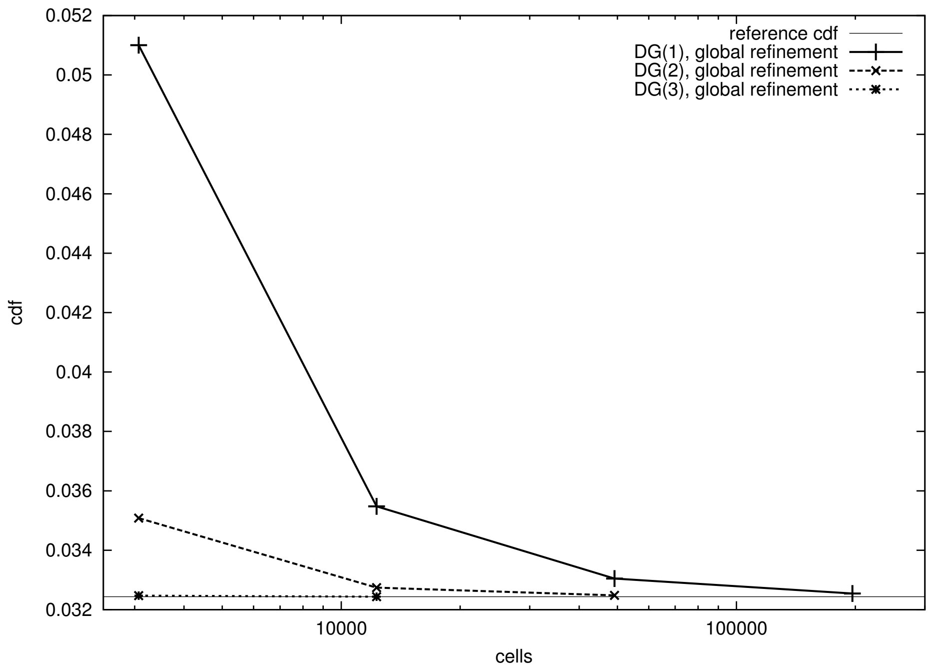


3.072 cells
196.608 dof
8 min

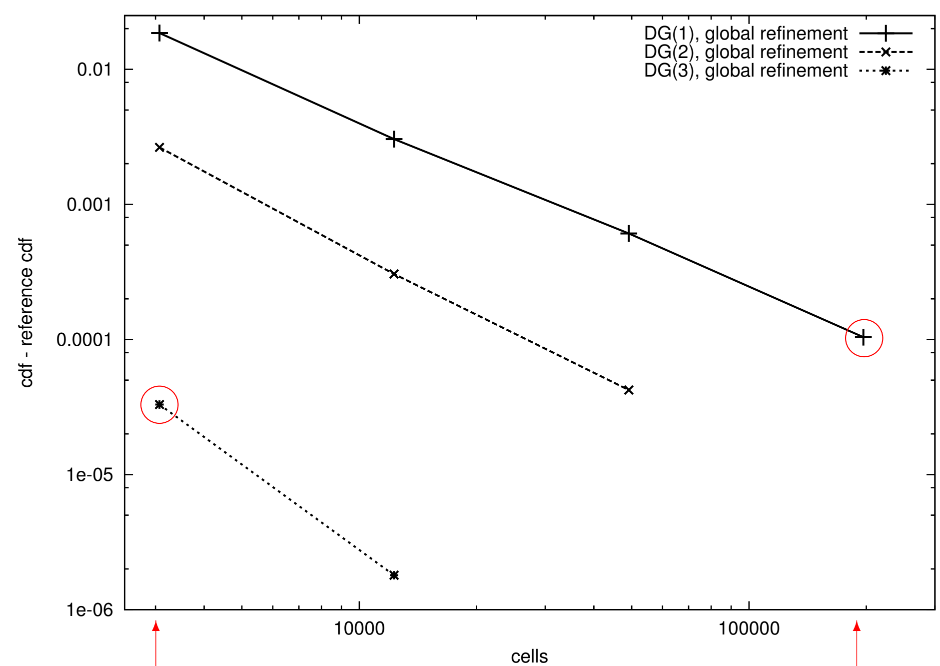
196.608 cells
3.145.728 dof
136 min

Convergence of cdf

cdf



cdf - reference cdf



3.072 cells
196.608 dof
8 min

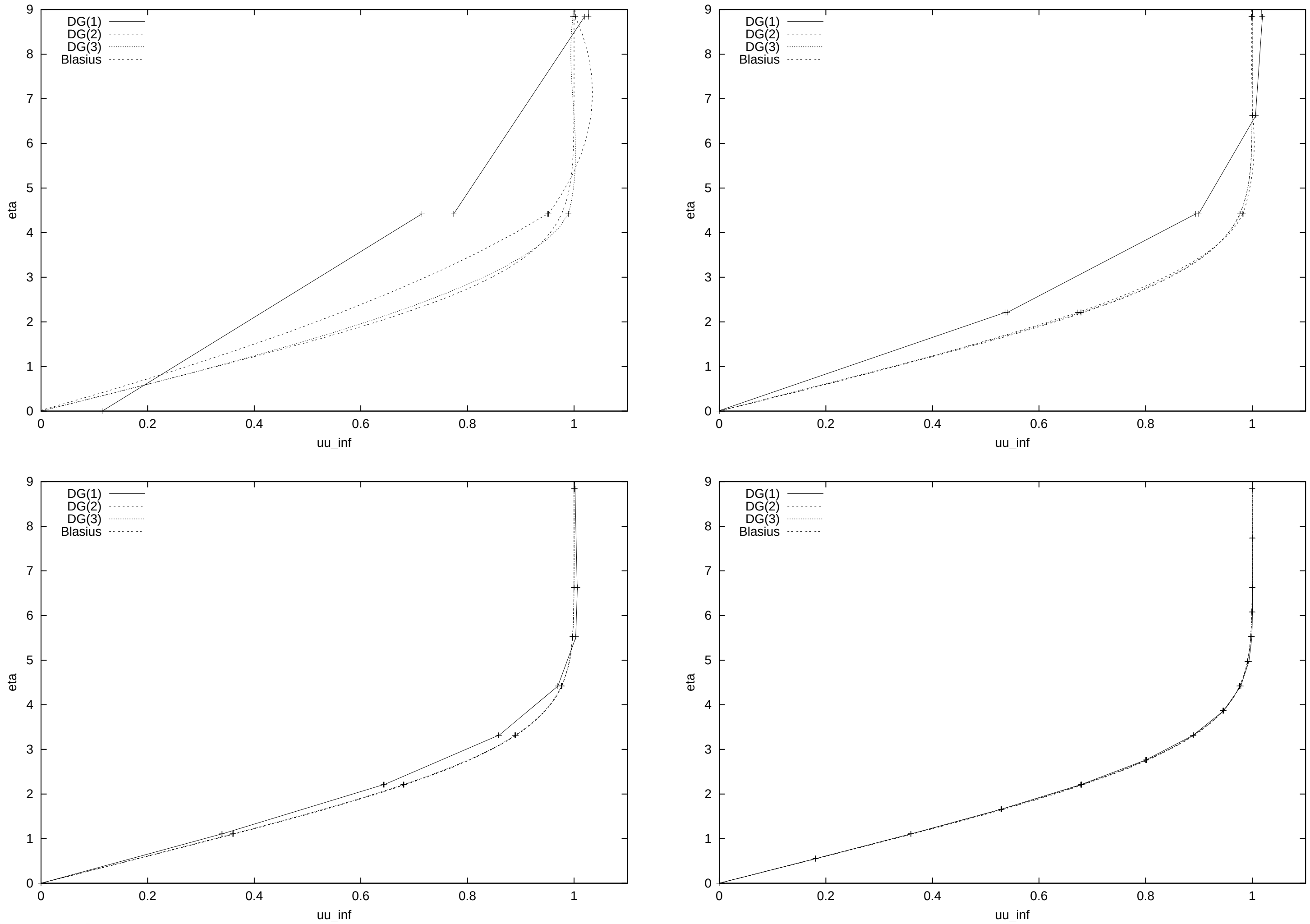
196.608 cells
3.145.728 dof
136 min

786.432 cells
12.582.912 dofs
≈ 12 h (extrapolated)

Higher order approximation of viscous boundary layers

Higher order approximation of viscous boundary layers

Flat plate problem: $M = 0.01$, $Re = 10000$, see [Hartmann,Houston2006]



Higher order approximation of viscous boundary layers

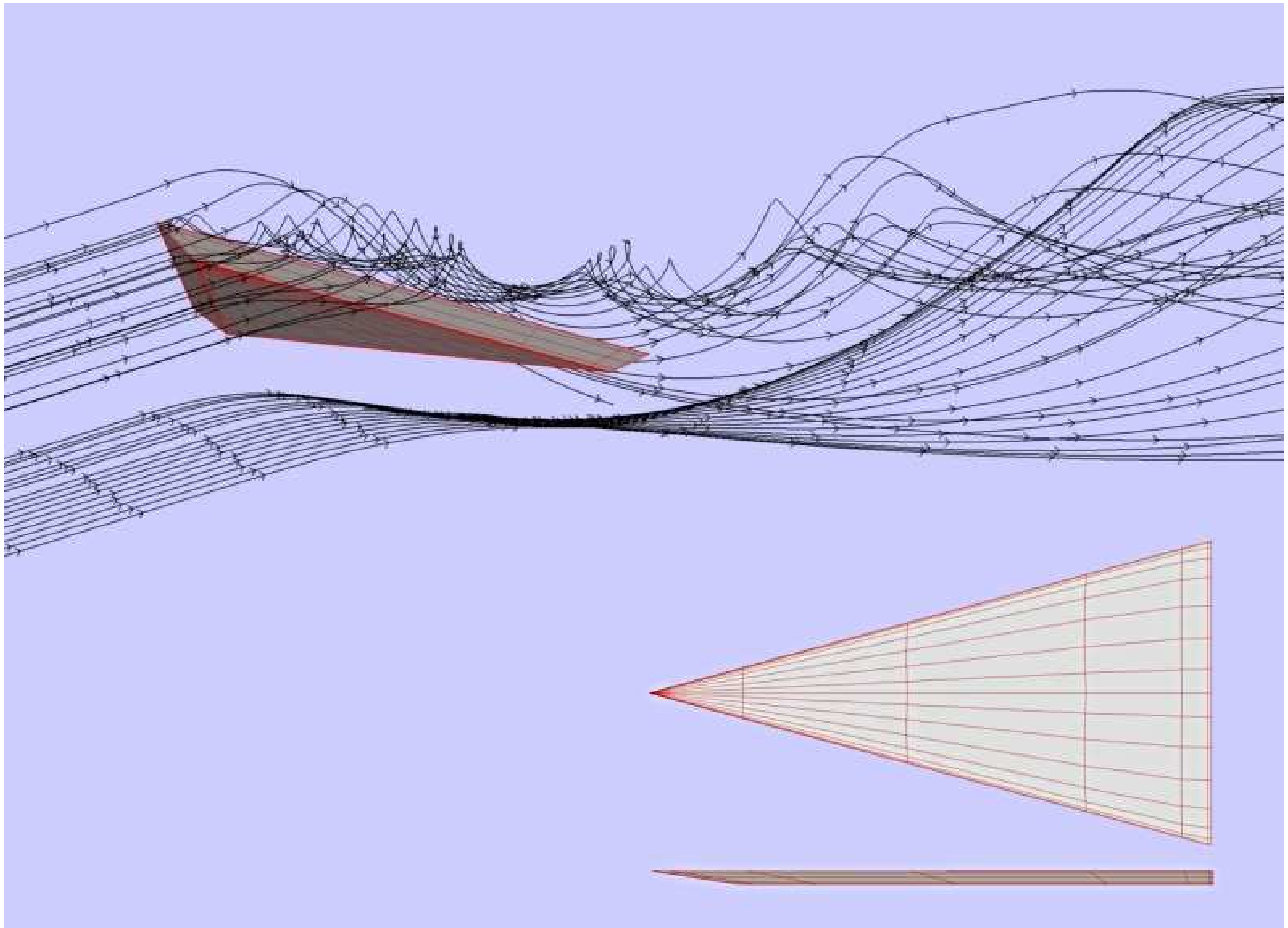
Flat plate problem: $M = 0.01$, $Re = 10000$

Approximation of viscous force exerted on wall up to 5% requires

	DG(1)	DG(2)	DG(3)
elements	36	5	3
DoF	72	15	12

orthogonal to the wall

Laminar Delta Wing

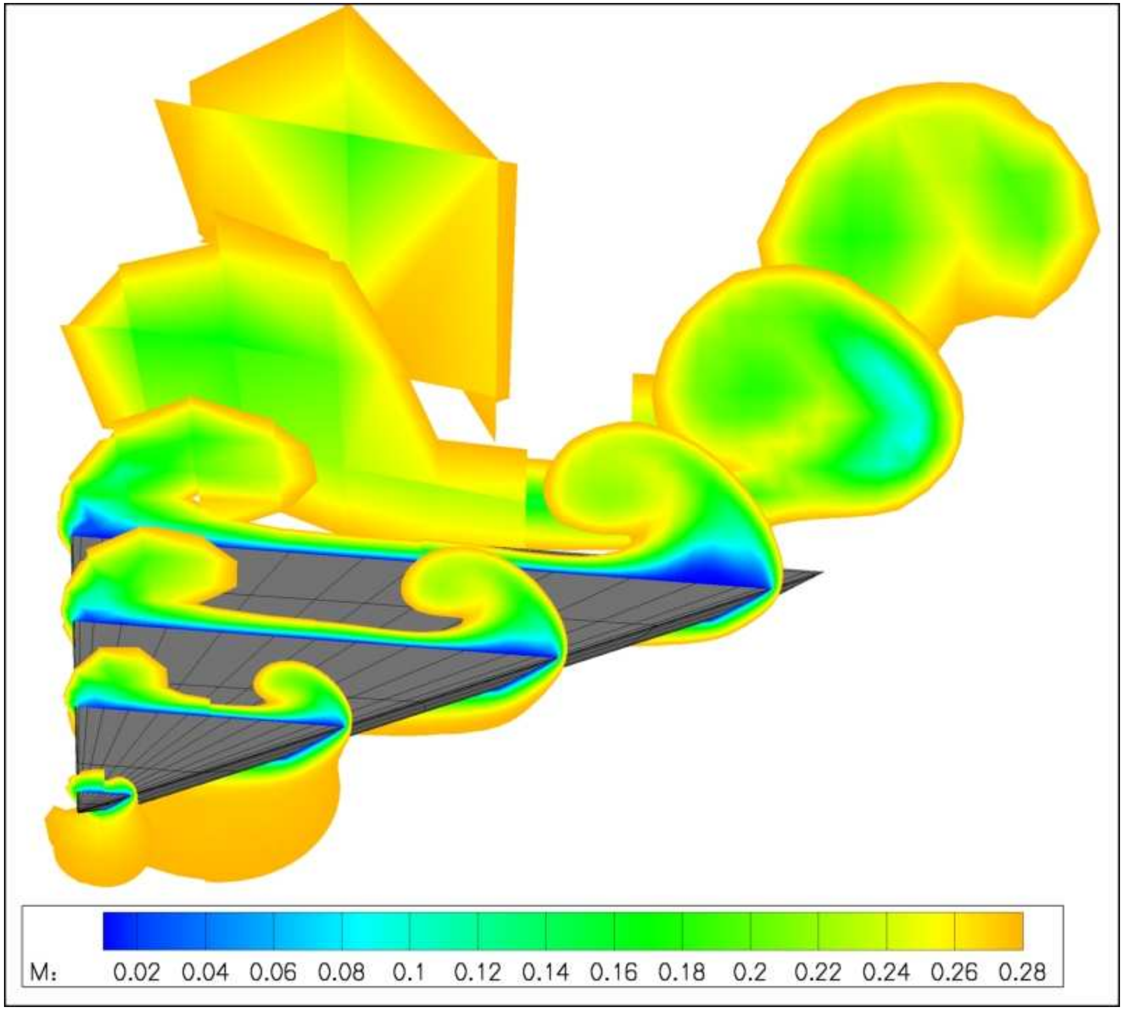


ADIGMA BTC-2 Test case

$M = 0.3$,
$\alpha = 12.5^\circ$,
$Re = 4000$,

isothermal noslip wall boundary

Laminar Delta Wing

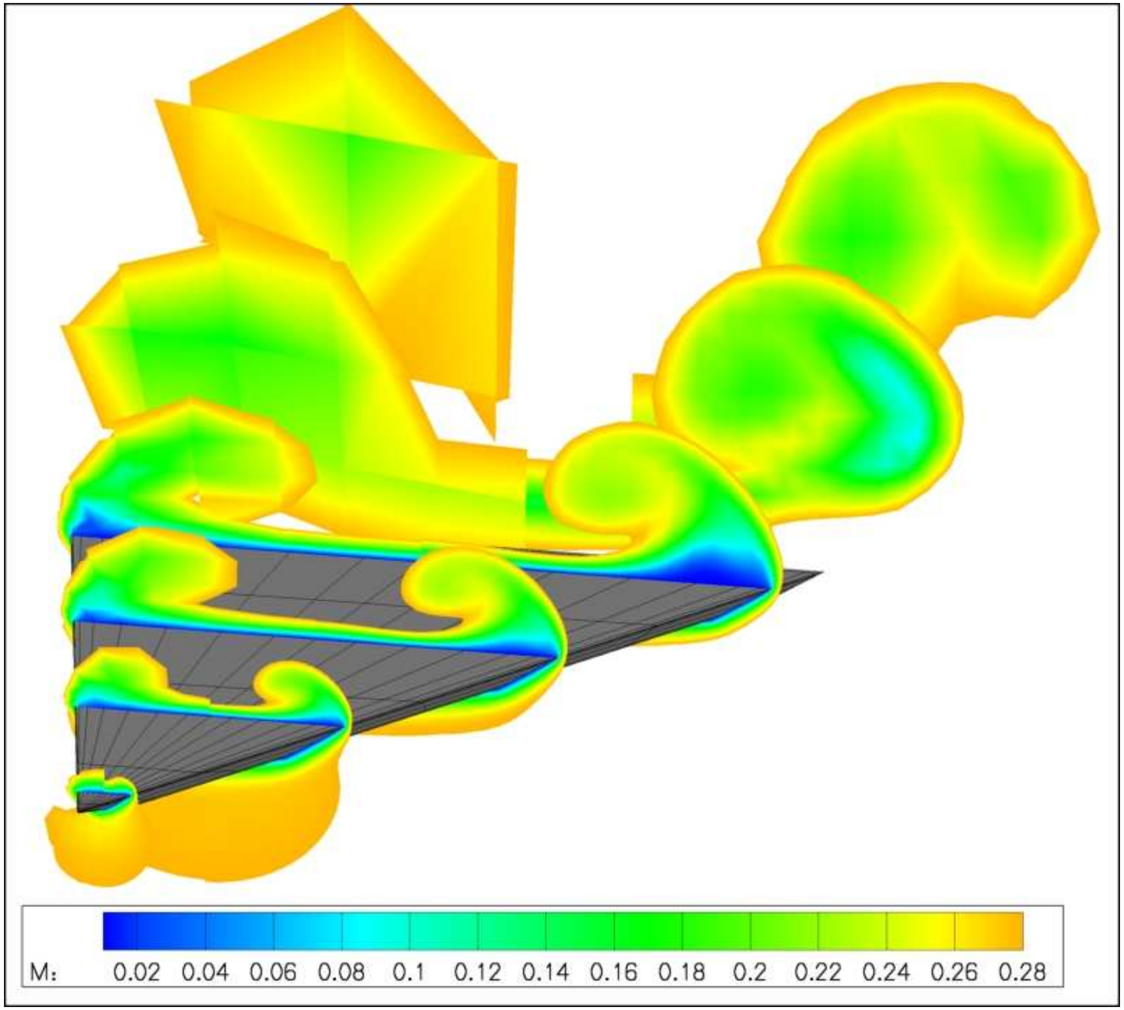


$M = 0.3, \alpha = 12.5^\circ,$
$Re = 4000$

3264 elements on the half domain

left: DG(1), 2nd order
right: DG(4), 5th order

Laminar Delta Wing



$M = 0.3, \alpha = 12.5^\circ,$
$Re = 4000$

3264 elements on the half domain

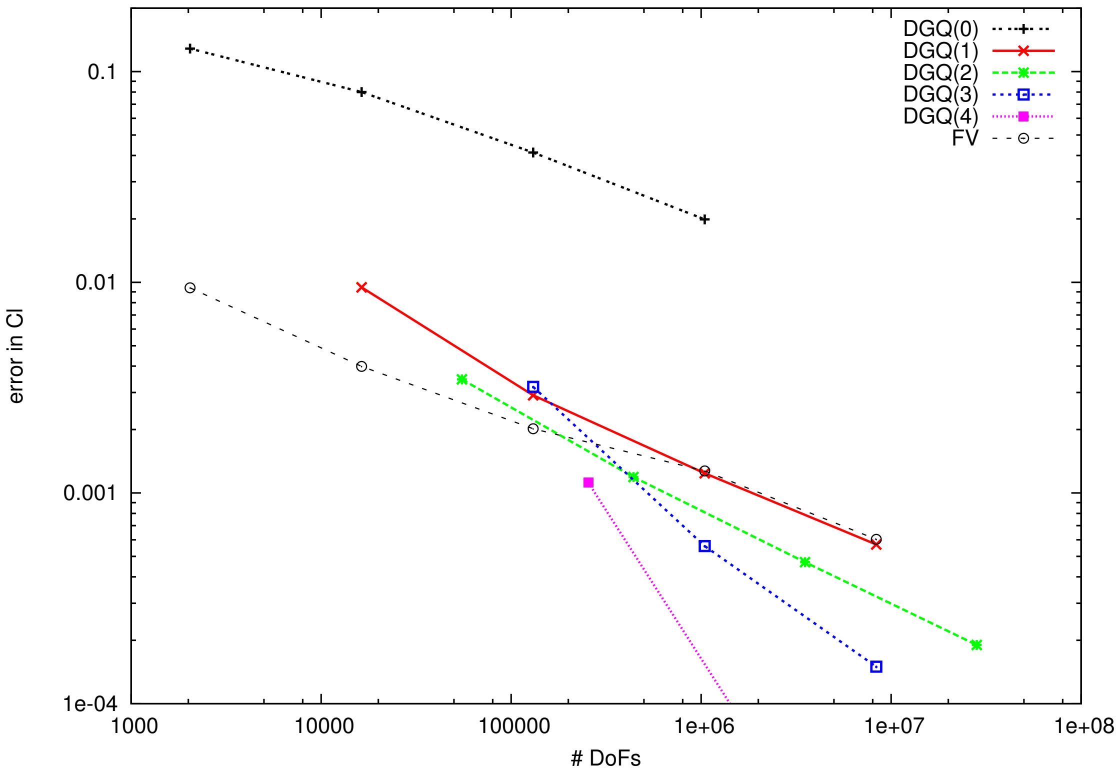
left: DG(1), 2nd order
right: DG(4), 5th order

DG(1), 40 DoFs/cell:
130,560 DoFs

DG(4), 625 dofs/cell:
2,040,000 DoFs

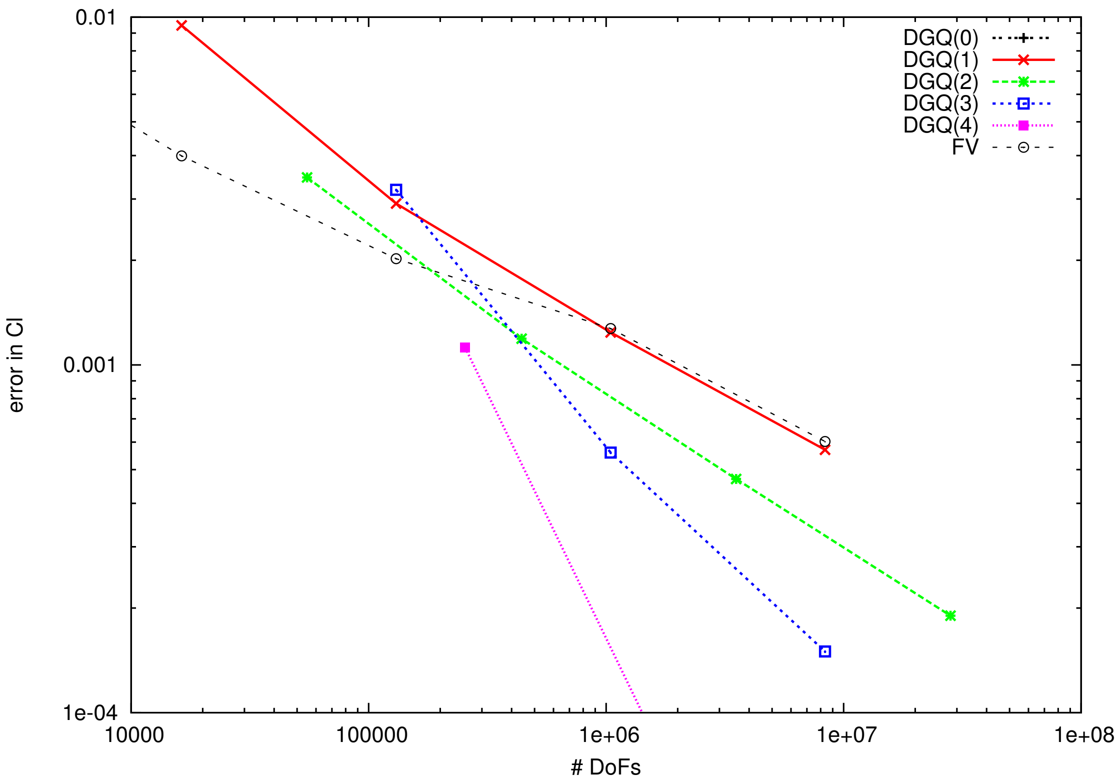
Laminar Delta Wing, $M = 0.3, \alpha = 12.5°, Re = 4000$

Convergence of c_l on a sequence of globally refined (non-nested) meshes.



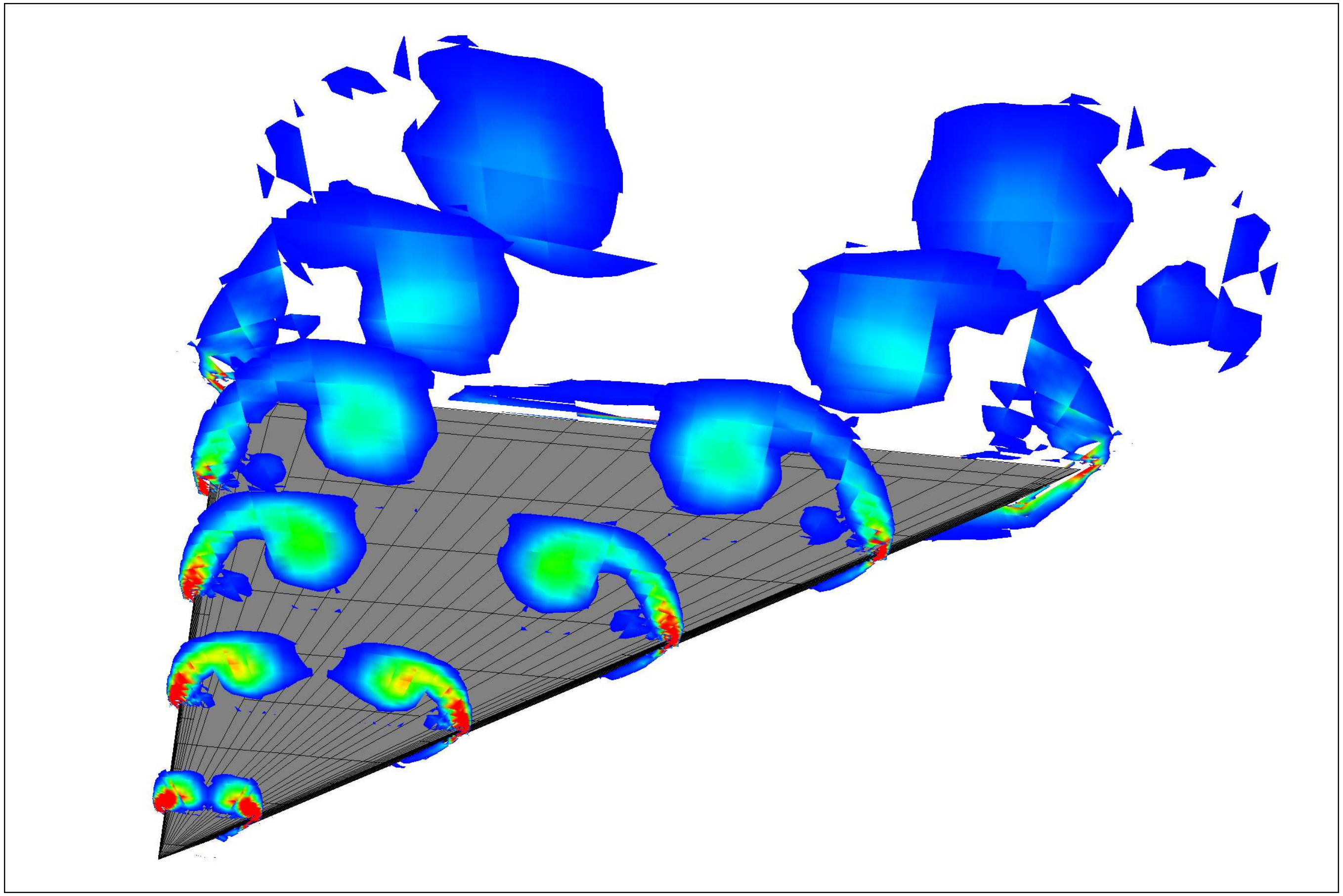
Laminar Delta Wing, $M = 0.3, \alpha = 12.5°, Re = 4000$

Convergence of c_l on a sequence of globally refined (non-nested) meshes.



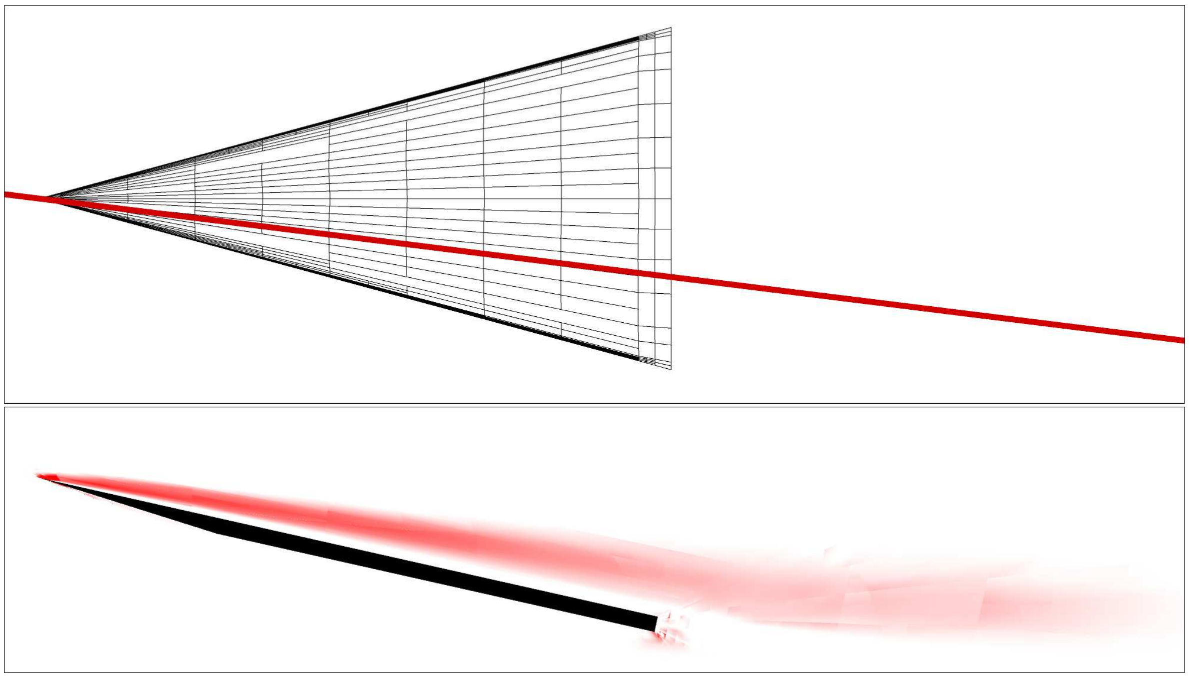
Laminar Delta Wing

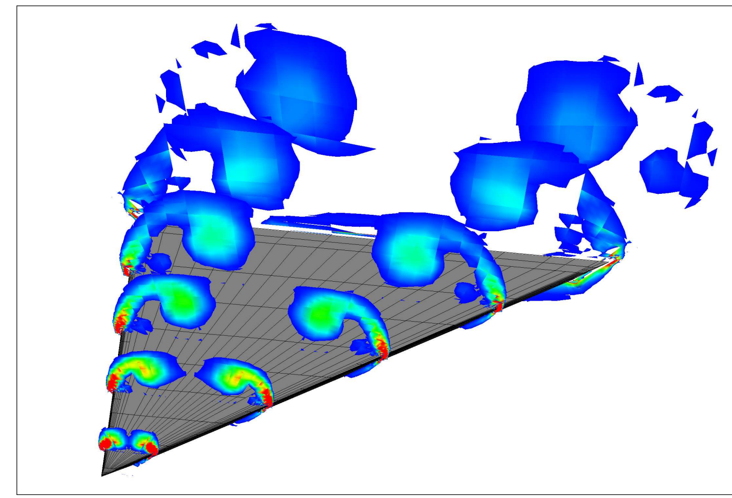
DG(3) discretization on locally refined grid of 8.122 elements (half domain)



Delta Wing, laminar flow, $M = 0.3, \alpha = 12.5°, Re = 4000$

Cut along one of the vortices

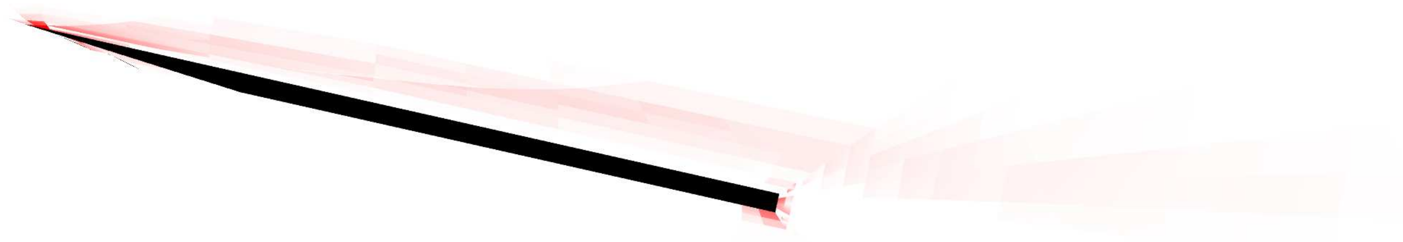




Delta Wing, laminar flow, $M = 0.3, \alpha = 12.5^\circ, Re = 4000$

DG discretization on locally refined grid of 8.122 elements (half domain)

p=1



p=2

p=3

Error estimation

Error estimation for single target quantities

Given a discretization: find $\mathrm{u}_h \in \mathrm{V}_h$ such that

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{1}$$

and a target quantity J.
Computed: $J(\mathrm{u}_h)$, exact (but unknown): $J(\mathrm{u})$, what is $J(\mathrm{u}) - J(\mathrm{u}_h)$?!

By employing a duality argument obtain (approximate) error representation

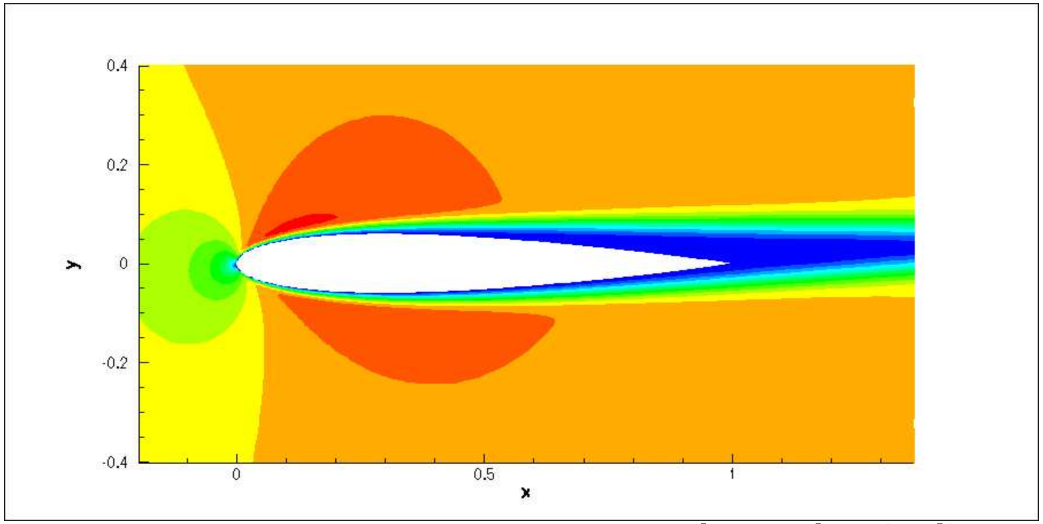
$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{u}_h) &= -\mathcal{N}(\mathbf{u}_h, \mathbf{z}) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}) \\ &\approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa =: \eta, \end{aligned}$$

Discrete adjoint problem: find $\tilde{\mathrm{z}}_h \in \tilde{\mathrm{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_h) = J'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h.$$

Error estimation for single target quantities: Example

ADIGMA MTC-3,
laminar flow,
$M = 0.5, \alpha = 2^\circ,$
$Re = 5000$



Mach number isolines

Force coefficients:		**required accuracy**
pressure induced drag coefficient:	$J(\mathrm{u}) = c_{dp}$	**5e-4**
viscous drag coefficient:	$J(\mathrm{u}) = c_{df}$	**5e-4**
total lift coefficient:	$J(\mathrm{u}) = c_l$	**5e-3**
total moment coefficient:	$J(\mathrm{u}) = c_m$	**5e-4**

Error estimation for single target quantity: $J(\mathrm{u}) = c_{dp}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathrm{u}) = c_{dp}$ **(pressure induced drag), Ref.value:** $J_{cdp}^{ref}(\mathrm{u}) = 0.02380$

		error in c_{dp}		
cells	DoFs	exact	estimate	ratio
400	6400	1.034e-03	-1.404e-03	-1.36
652	10432	3.341e-03	2.959e-03	0.89
1090	17440	4.045e-04	5.712e-04	1.41
1801	28816	-2.079e-04	-1.091e-04	0.52
3034	48544	-2.344e-04	-1.890e-04	0.81
5047	80752	-1.529e-04	-1.387e-04	0.91
8527	136432	-8.055e-05	-7.536e-05	0.94
14410	230560	-4.357e-05	-3.762e-05	0.86
24406	390496	-2.366e-05	-2.314e-05	0.98

Error estimation for single target quantity: $J(\mathrm{u}) = c_{df}$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathrm{u}) = c_{df}$ **(viscous drag), Ref.value:** $J_{cdf}^{ref}(\mathrm{u}) = 0.0322835$

		error in c_{df}		
cells	DoFs	exact	estimate	ratio
400	6400	1.076e-02	1.525e-02	1.42
655	10480	-2.973e-03	-2.592e-03	0.87
1093	17488	-1.415e-03	-1.418e-03	1.00
1804	28864	-3.947e-04	-4.326e-04	1.10
2989	47824	-9.136e-05	-1.116e-04	1.22
5110	81760	-3.787e-05	-4.518e-05	1.19
8476	135616	-1.919e-05	-2.071e-05	1.08
14185	226960	-1.319e-05	-1.619e-05	1.23
23638	378208	-1.048e-05	-1.052e-05	1.00

Error estimation for single target quantity: $J(\mathbf{u}) = c_l$

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_l$ **(total lift), Ref.value:** $J_{cl}^{ref}(\mathbf{u}) = 0.037286$

		error in c_l		
cells	DoFs	exact	estimate	ratio
400	6400	-1.175e-01	-5.867e-02	0.50
658	10528	6.548e-03	6.841e-03	1.04
1108	17728	-1.292e-03	-1.159e-03	0.90
1861	29776	-1.784e-03	-1.891e-03	1.06
3118	49888	-1.239e-03	-1.266e-03	1.02
5236	83776	-6.504e-04	-6.704e-04	1.03
8746	139936	-2.623e-04	-2.622e-04	1.00

Error estimation for single target quantity: $J(\mathbf{u}) = c_m$

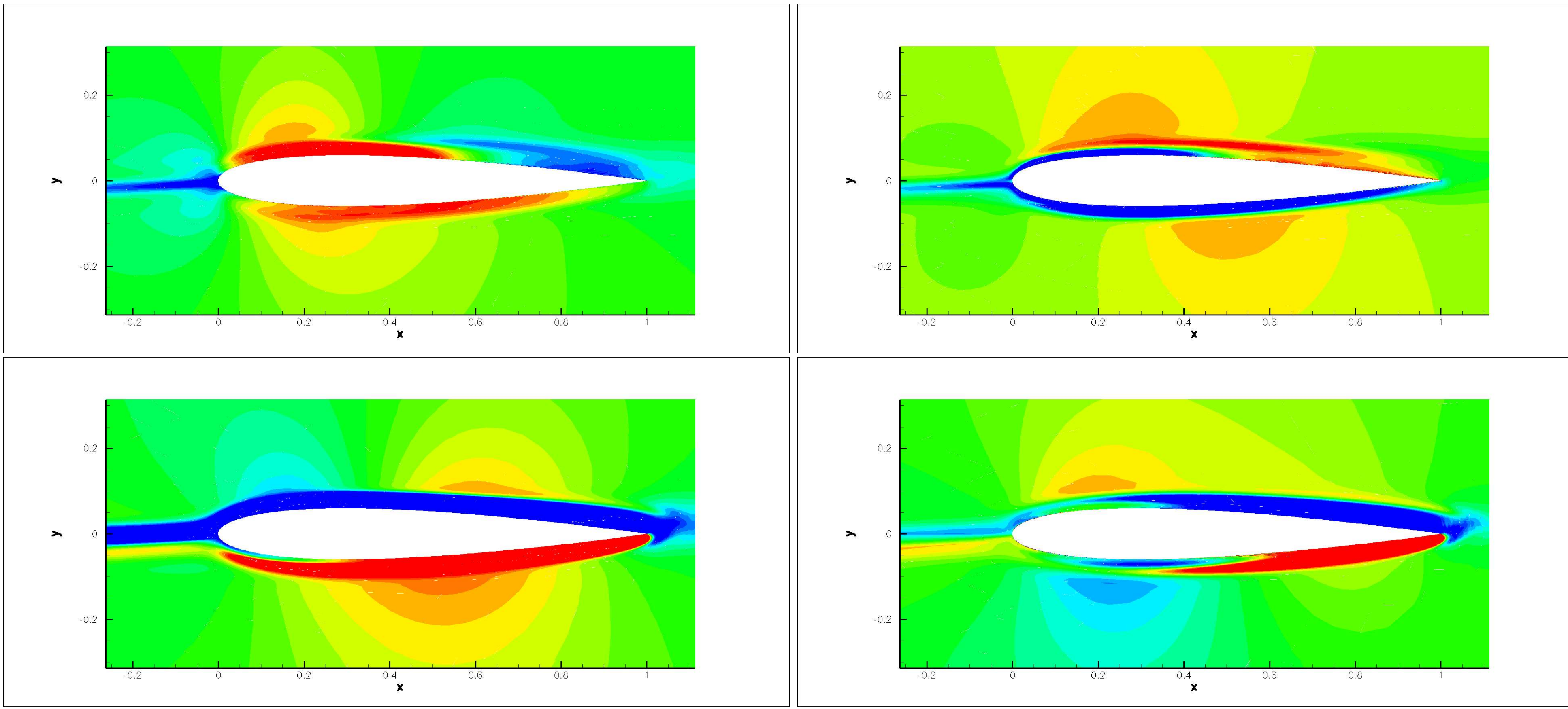
Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

Target quantity: $J(\mathbf{u}) = c_m$ **(total moment), Ref.value:** $J_{cm}^{ref}(\mathbf{u}) = -0.01661$

		error in c_m		
cells	DoFs	exact	estimate	ratio
400	6400	-1.221e-03	-3.035e-03	2.49
667	10672	2.883e-03	3.001e-03	1.04
1138	18208	3.862e-04	4.378e-04	1.13
1867	29872	9.083e-05	8.543e-05	0.94
3130	50080	6.199e-05	5.807e-05	0.94

Error estimation for single target quantities

z_1 **components of adjoint solutions.** **Top: cdp, cdf;** **Bottom: cl, cm.**



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Error estimation for multiple target quantities

Given N target quantities $J_i(\mathbf{u}), i = 1, \ldots, N$.

The *direct approach* requires N discrete adjoint problems: find $\tilde{\mathbf{z}}_{i,h} \in \tilde{\mathbf{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_{i,h}) = J_i'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h, \qquad i = 1, \ldots, N,$$

to obtain error estimates for N target quantities

$$J_i(\mathbf{u}) - J_i(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}_i) \approx \mathcal{R}(\mathbf{u}_h, \mathbf{z}_{i,h}), \qquad i = 1, \ldots, N,$$

The *new approach*, originally developed in [Hartmann,Houston2003] requires *one* discrete adjoint-adjoint problem (error equation): find $\tilde{\mathbf{e}}_h \in \tilde{\mathbf{V}}_h$ such that

$$\mathcal{N}'[\mathbf{u}_h](\tilde{\mathbf{e}}_h, \mathbf{w}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

to obtain error estimates for N target quantities

$$J_i(\mathbf{u}) - J_i(\mathbf{u}_h) \approx J_i'[\mathbf{u}_h](\mathbf{e}) \approx J_i'[\mathbf{u}_h](\tilde{\mathbf{e}}_h), \qquad i = 1, \ldots, N,$$

Error estimation for multiple target quantities

Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$

On each mesh compute primal solution u_h **and adjoint-adjoint solution** $\tilde{\mathrm{e}}_h$**.**

Evaluate *exact* error: $J_i^{\text{ref}}(\mathbf{u}) - J_i(\mathbf{u}_h), \quad i = 1, \ldots, N,$

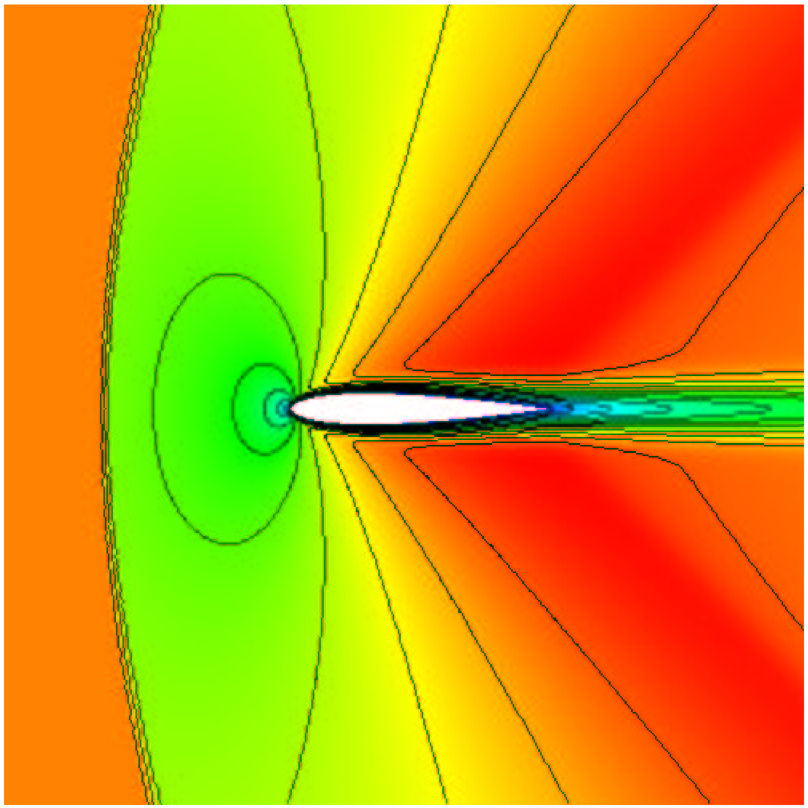
Evaluate error *estimate*: $J_i'[\mathrm{u}_h](\tilde{\mathrm{e}}_h), \quad i = 1, \ldots, N,$

	error in cdp		error in cdf		error in cl		error in cm	
#cells	exact	estimate	exact	estimate	exact	estimate	exact	estimate
400	1.03e-03	-2.92e-03	1.08e-02	1.62e-02	-1.18e-01	-6.59e-02	-1.22e-03	-4.36e-03
655	1.39e-03	1.38e-03	-3.02e-03	-2.89e-03	6.30e-03	4.15e-03	2.99e-03	2.67e-03
1111	-1.04e-04	8.65e-05	-1.42e-03	-1.89e-03	-8.30e-04	-6.54e-04	4.76e-04	5.11e-04
1843	-6.28e-04	-5.28e-04	-5.20e-04	-6.46e-04	-1.83e-03	-1.91e-03	6.25e-05	3.49e-05
3061	-3.96e-04	-3.51e-04	-1.61e-04	-2.25e-04	-7.34e-04	-7.69e-04	3.15e-05	3.67e-05
5146	-1.82e-04	-1.63e-04	-9.03e-05	-1.11e-04	-4.86e-04	-3.94e-04	1.09e-05	1.35e-05

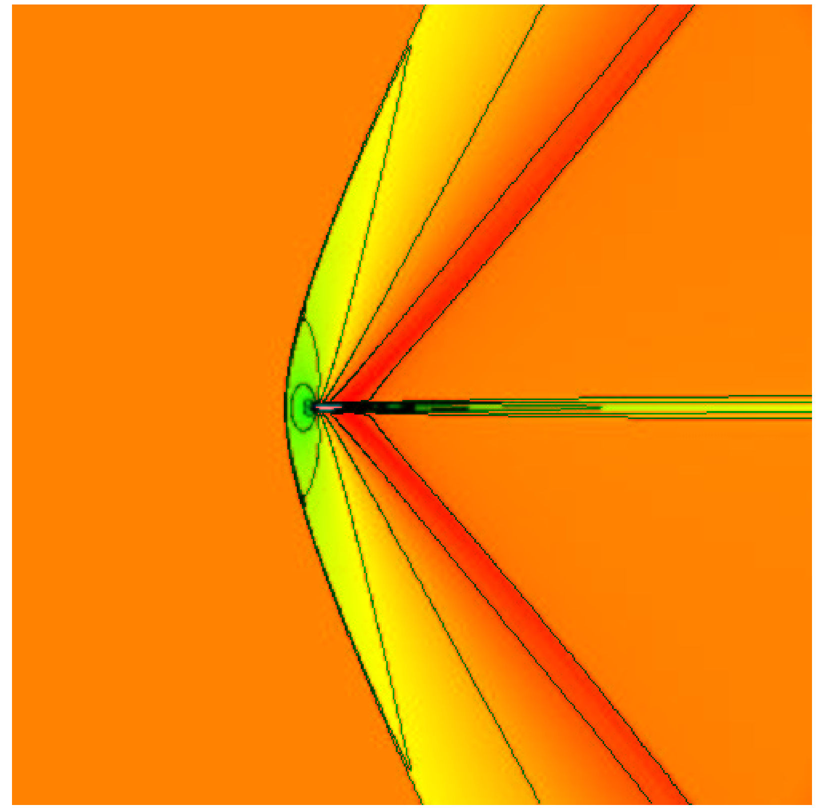
Goal-oriented (adjoint-based) refinement

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



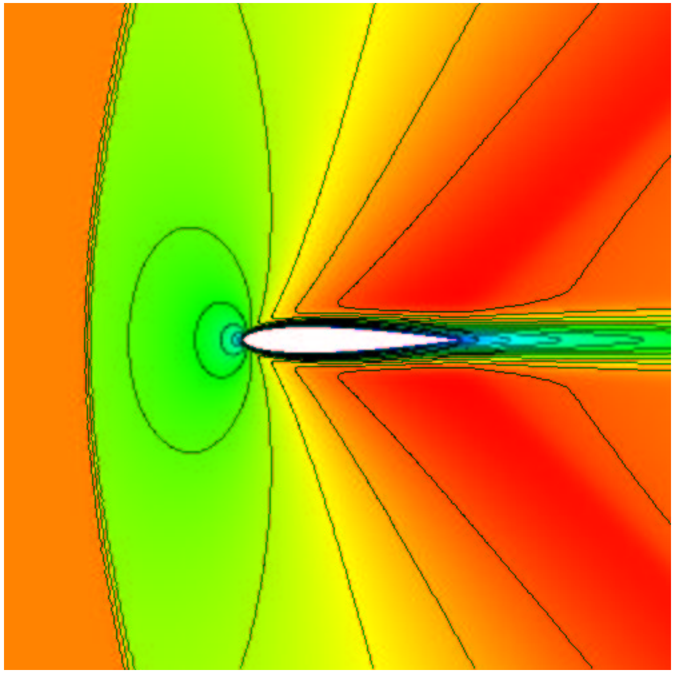
Mach isolines: close view



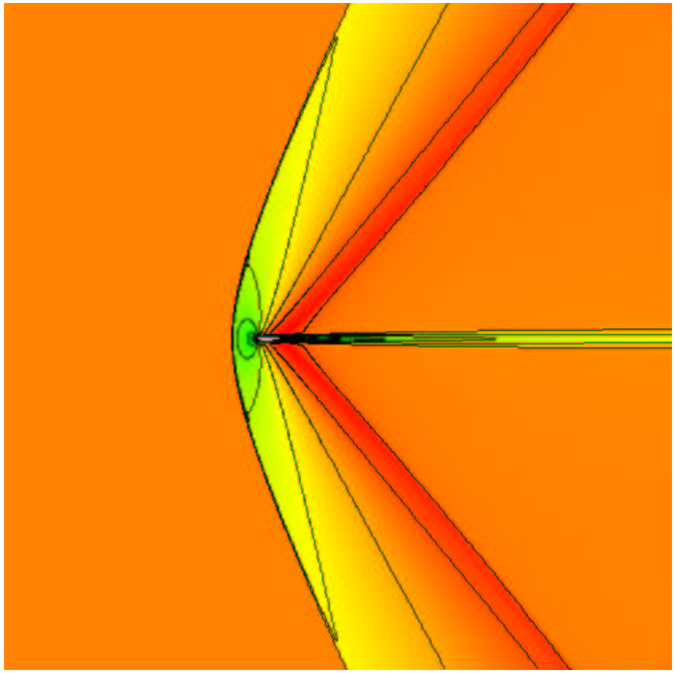
Mach isolines: distant view

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



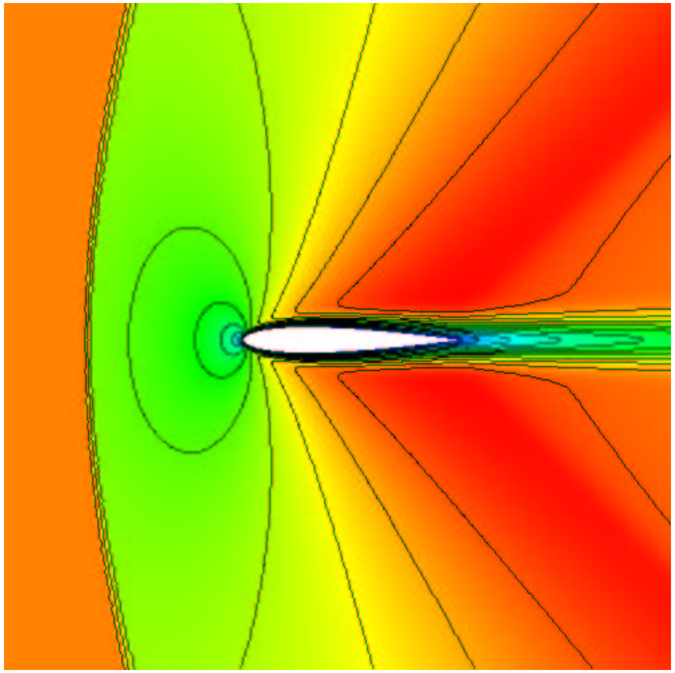
Mach isolines: close view



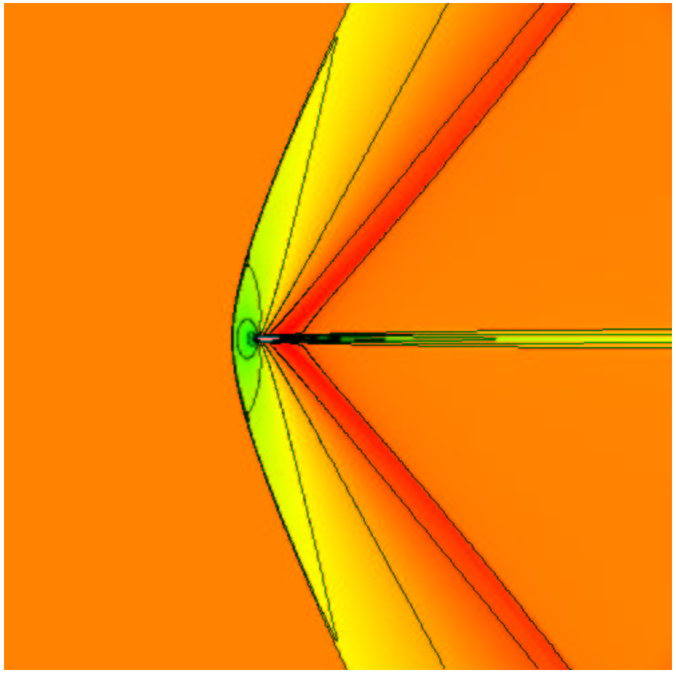
Mach isolines: distant view

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



Mach isolines: close view

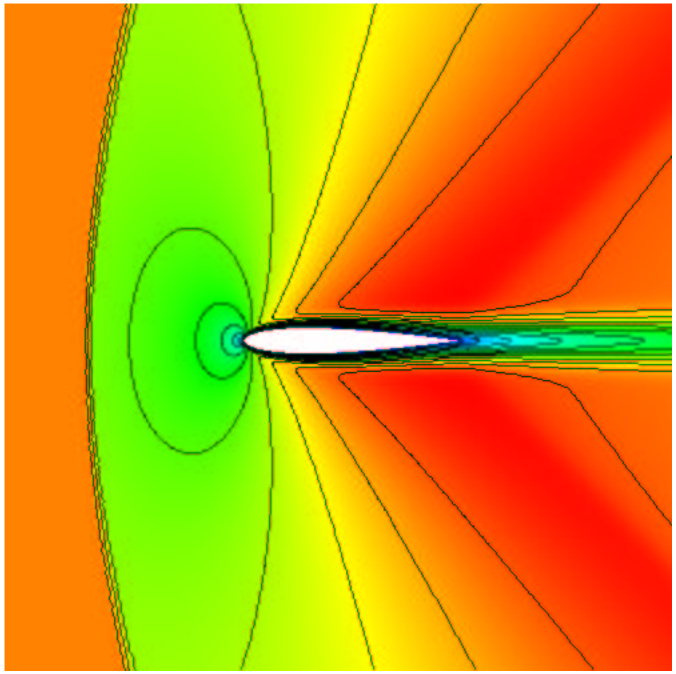


Mach isolines: distant view

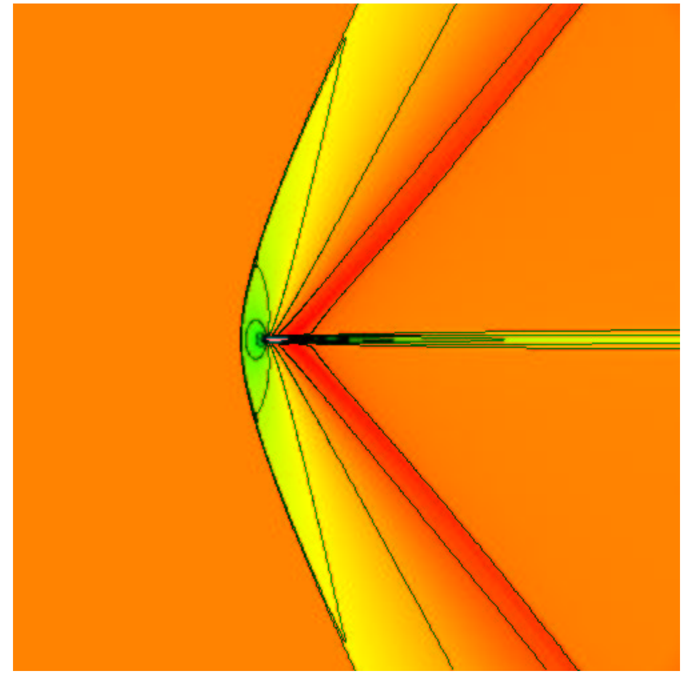
For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

Goal-oriented refinement

Viscous flow at $M = 1.2$, $Re = 1000$, $\alpha = 0$ around NACA0012 airfoil



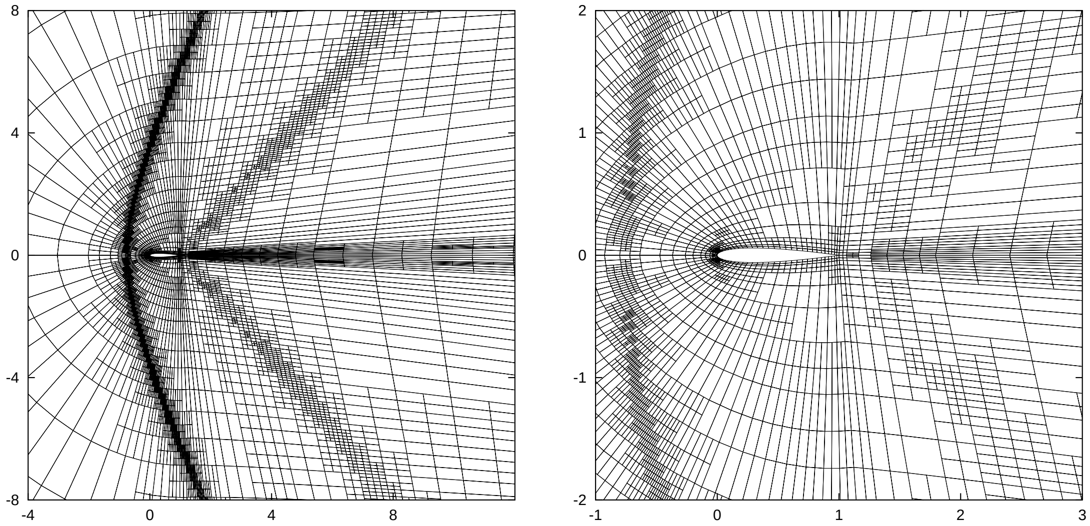
Mach isolines: close view

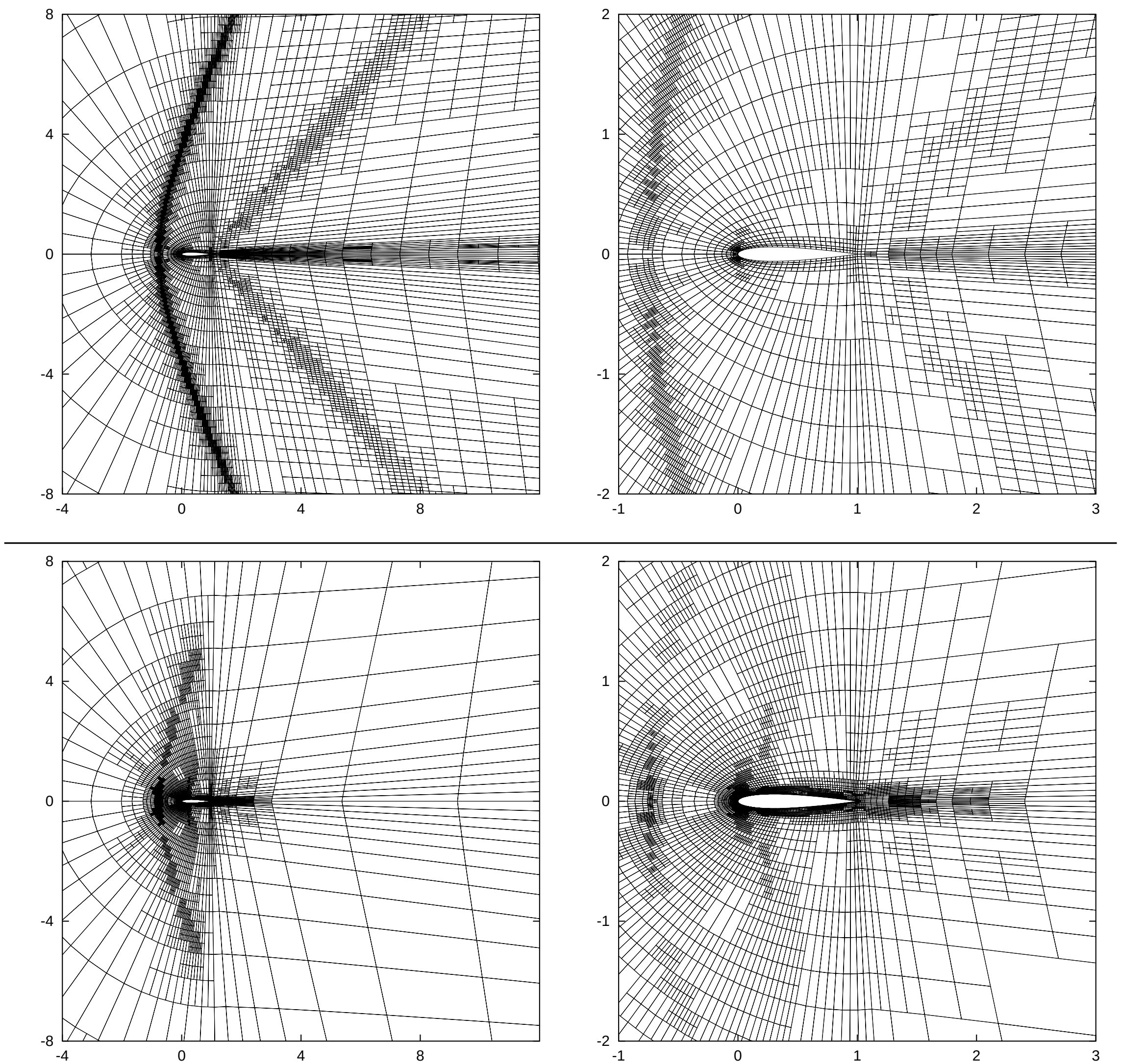


Mach isolines: distant view

For the efficient and accurate approximation of $J(\mathrm{u}) = c_{dp}$:

How should the mesh look like?


8
4
0
-4
-8
-4
0
4
8
2
1
0
-1
-2
-1
0
1
2
3


8
4
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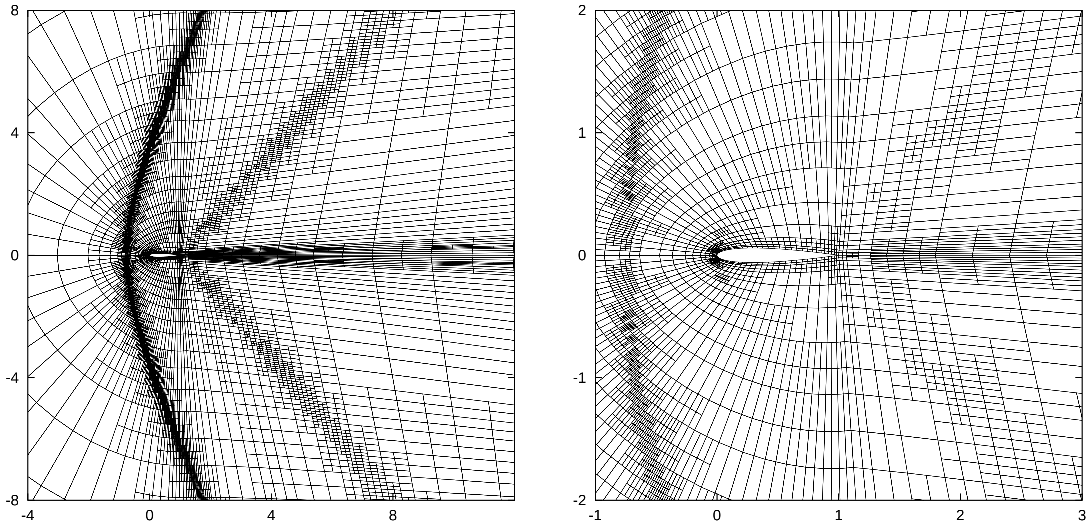
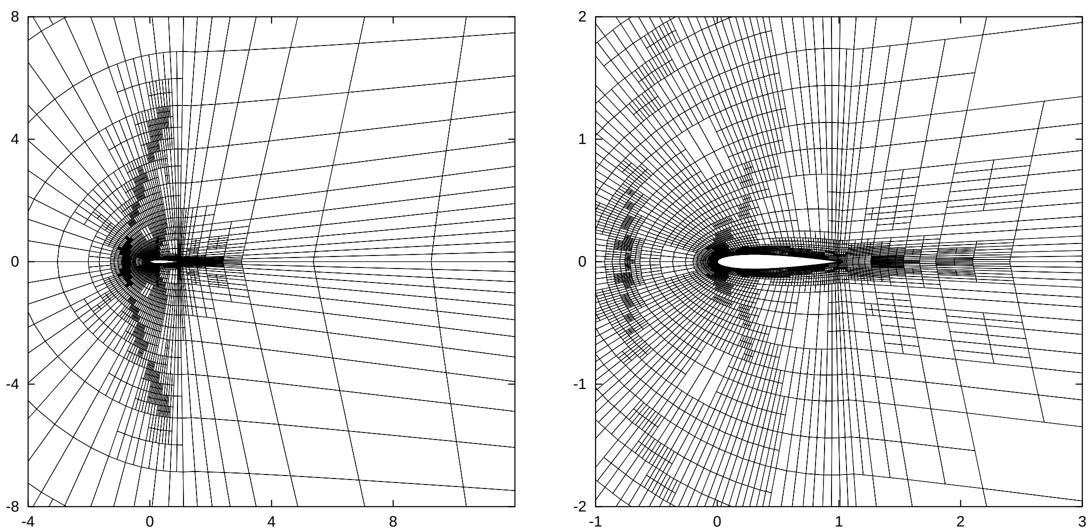
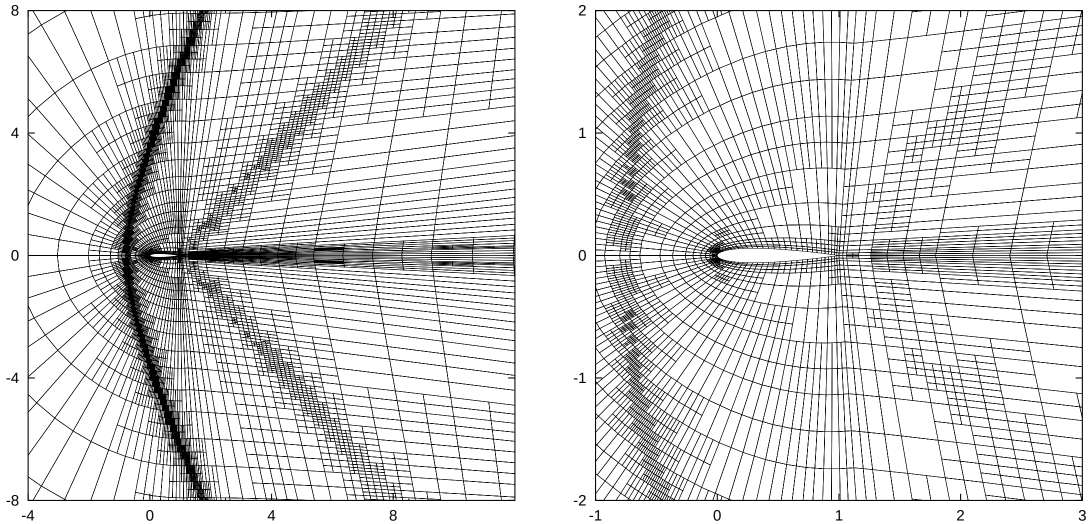




Residual-based refinement:
17670 cells with 282720 dofs
error in $c_{dp} : 1.9 \cdot 10^{-3}$
error in $c_{df} : 1.1 \cdot 10^{-2}$



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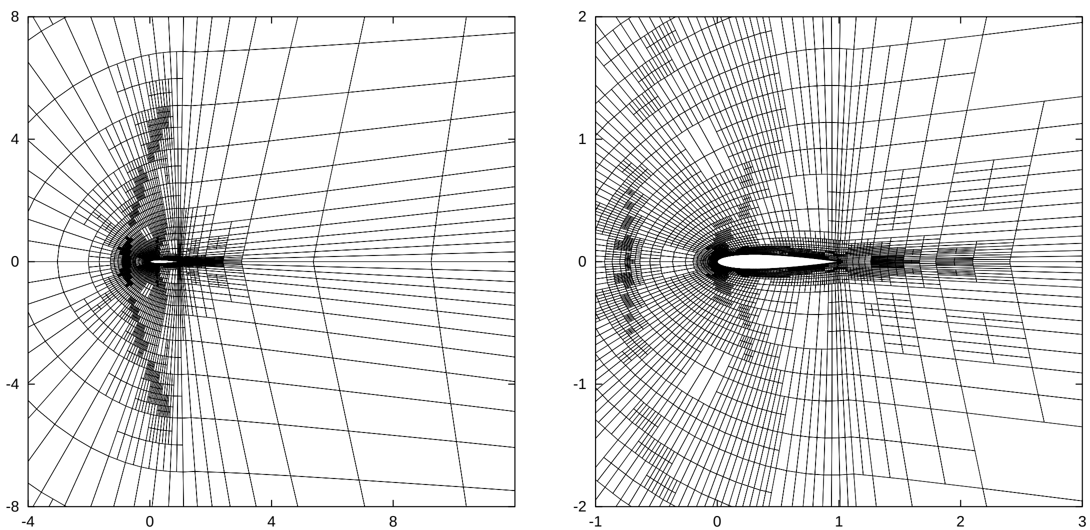


Goal-oriented refinement:
10038 cells with 160608 dofs
error in c_{dp} : $1.6 \cdot 10^{-4}$
error in c_{df} : $7.2 \cdot 10^{-4}$

First summary

Higher order discontinuous Galerkin discretization of the compressible Euler and Navier-Stokes equations

- Same accuracy on coarser meshes & less computational time than for 2nd order
- Accurate error estimation with respect to target quantities
- Efficient adjoint-based (goal-oriented) adaptive mesh refinement

Anisotropic refinement

Use a residual-based or adjoint-based indicator to select the elements to be refined

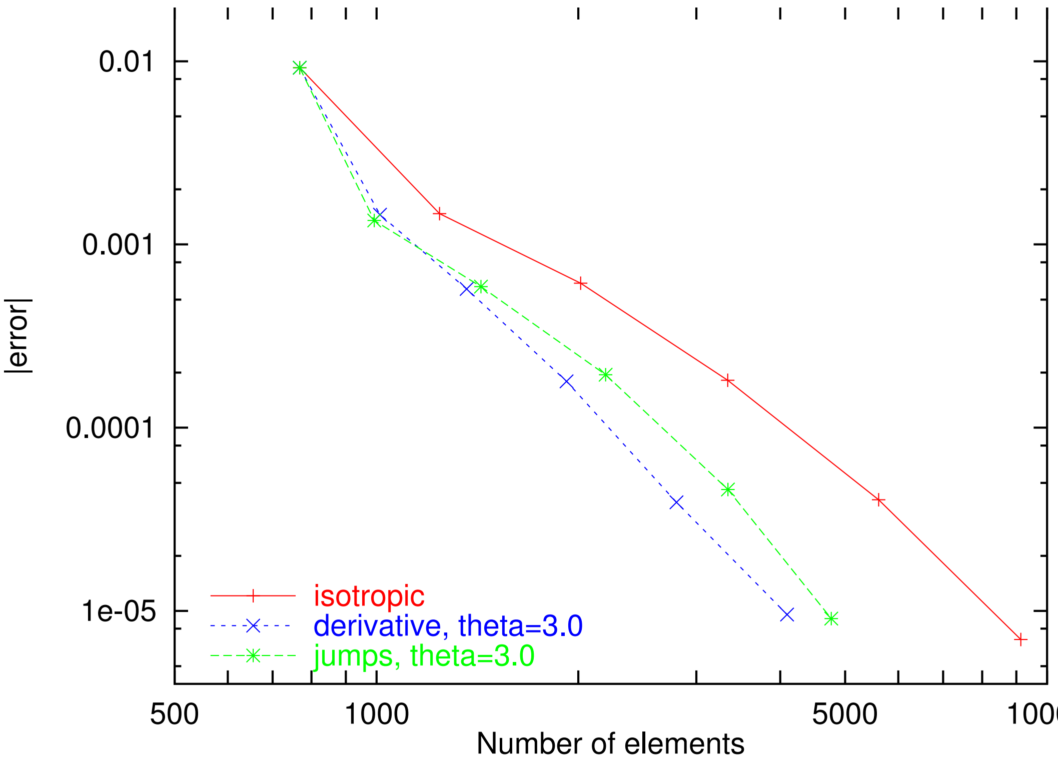
Use an anisotropic indicator to determine the anisotropic refinement case

Anisotropic indicators:

- Jump indicator: jump across a face is connected to approximation quality orthogonal to the face
- Derivative indicator: Hessian for 2nd scheme, higher order derivatives for higher order schemes

Anisotropic refinement: Laminar test case

$M = 0.5$, $Re = 5000$, $\alpha = 0$ flow around the NACA0012 airfoil



DG(2), i.e. 3rd order,
with adjoint-based refinement,
error measured in c_{dp}

Comparison of:

- **isotropic refinement**
- **anisotropic jump indicator**
- **anisotropic derivative indicator (3rd derivatives)**

Anisotropic refinement: First summary

- **Resolution of anisotropic flow features like shocks and boundary layers**
- **In combination with error estimation and goal-oriented (adjoint-based) refinement: Automatic generation of "optimal" initial layer spacing**

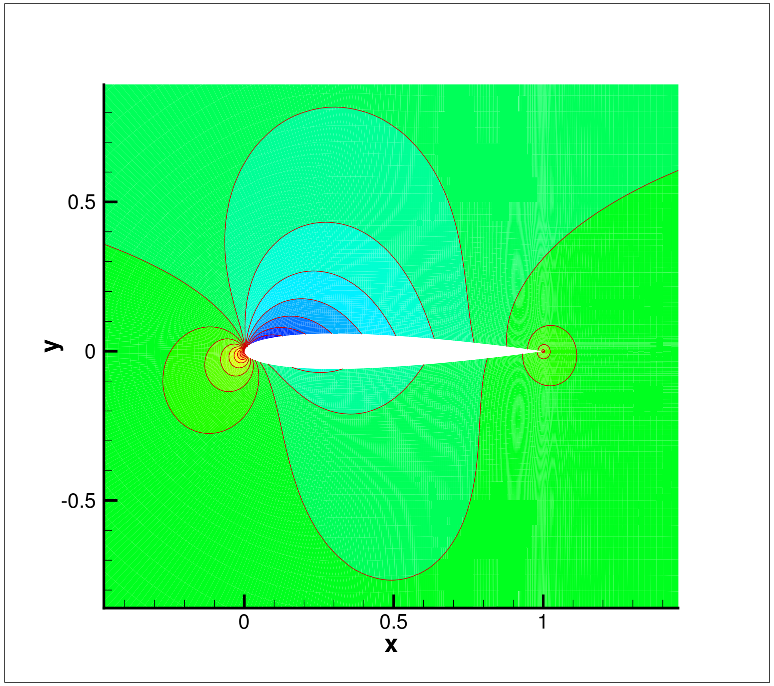
Anisotropic refinement: First summary

- Resolution of anisotropic flow features like shocks and boundary layers
- In combination with error estimation and goal-oriented (adjoint-based) refinement: Automatic generation of “optimal” initial layer spacing

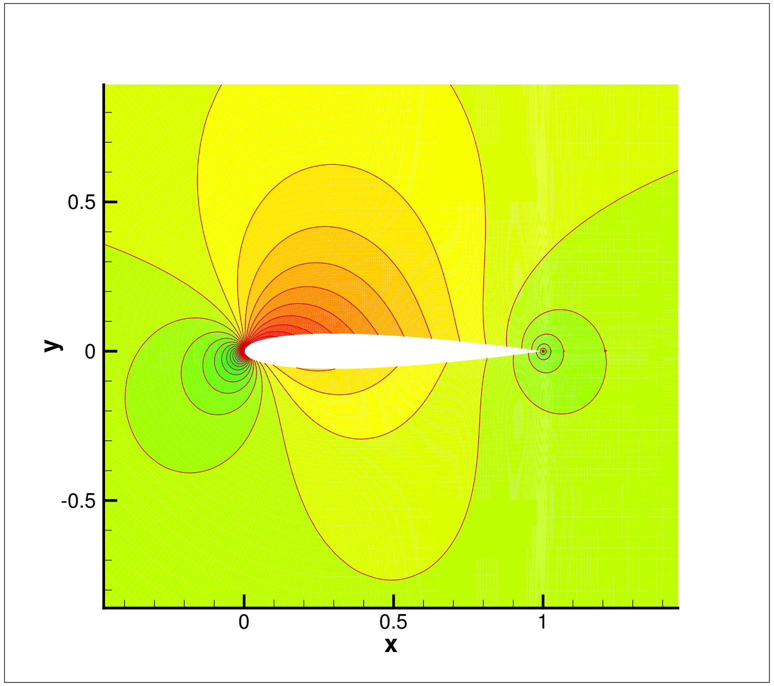
Furthermore:
Use anisotropic refinement to optimize grids with inappropriate/bad aspect ratios.

Anisotropic refinement applied to ADIGMA Test Case 1

compressible Euler equations: $M = 0.5, \alpha = 2$:



pressure

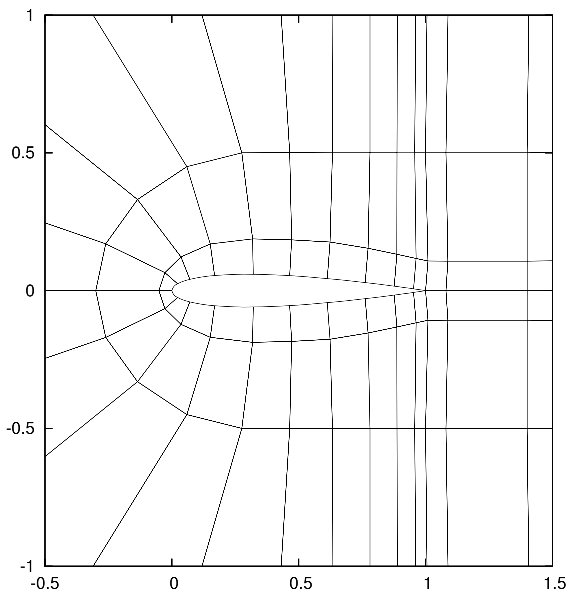


Mach number

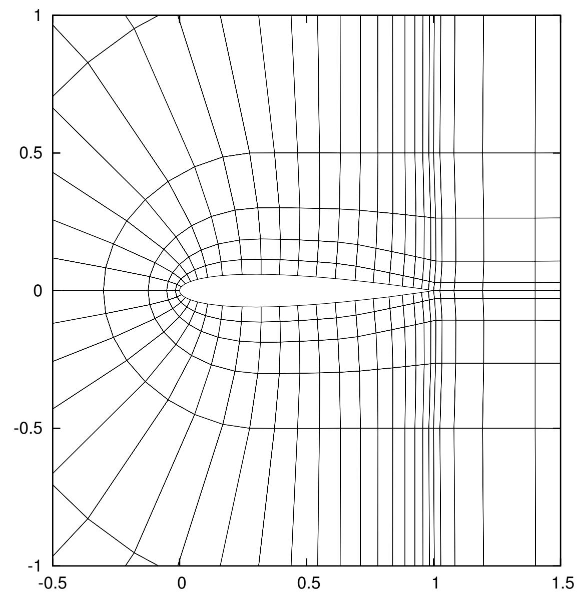
No anisotropic features

Test case 1: Euler, M=0.5, $\alpha = 2^\circ$

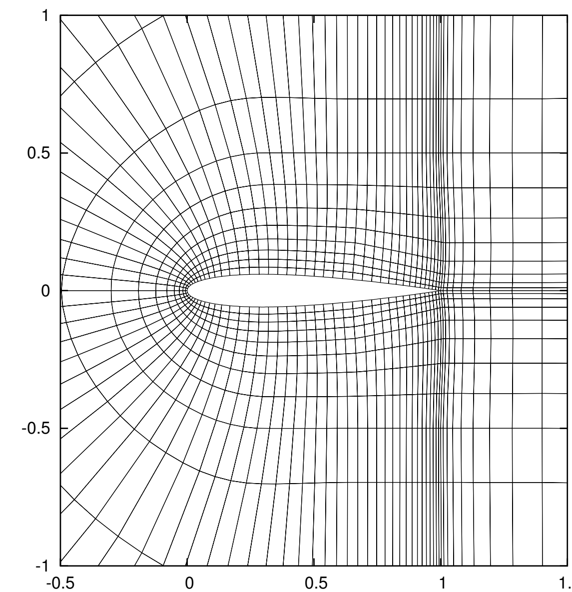
Grids: MTC_v1.3_NACA0012_INVISCID (by DLR)



level 6: 112 elements



level 5: 448 elements

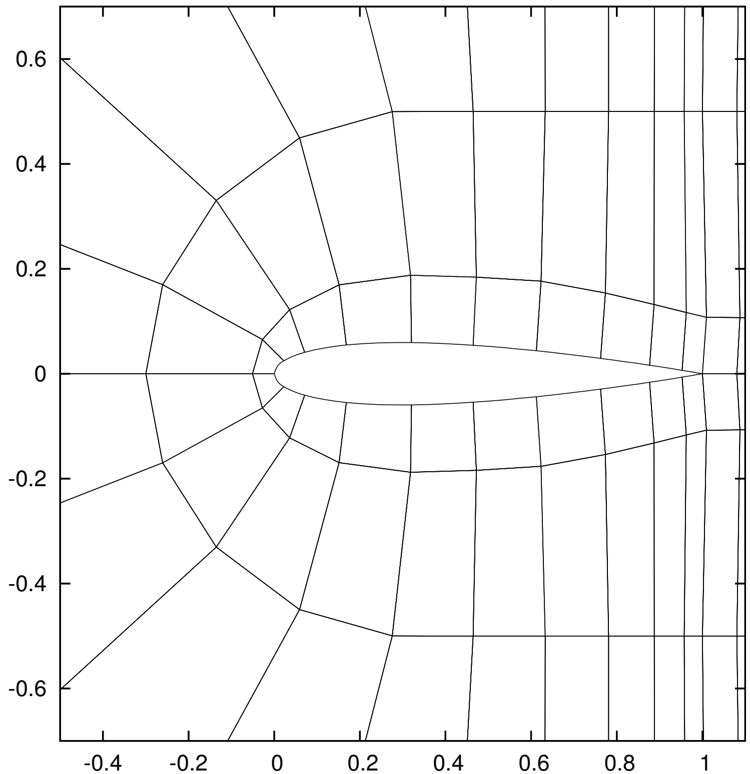


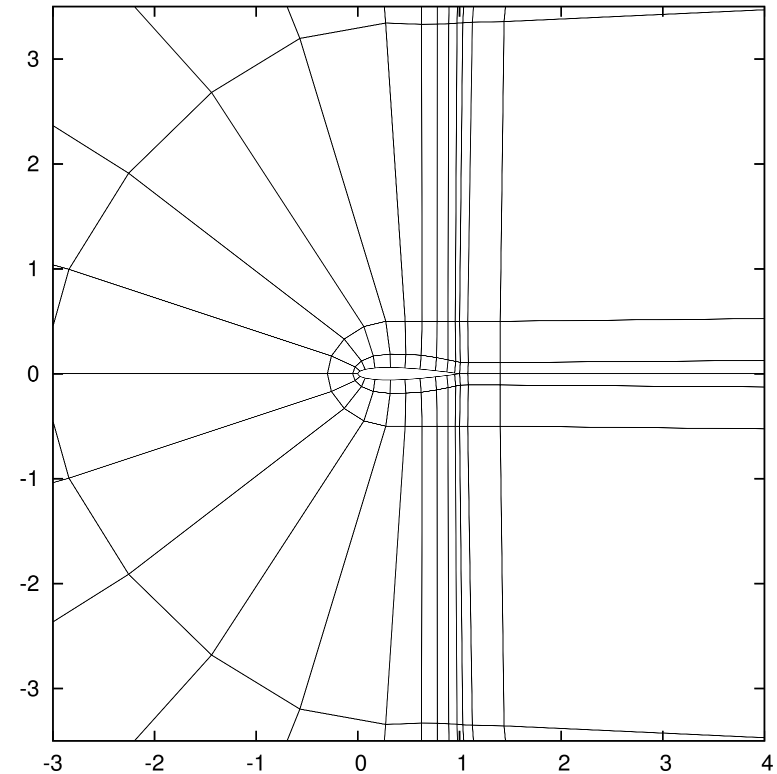
level 4: 1792 elements **...**

level 3: 7168 elements
level 2: 28672 elements
level 1: 114688 elements

Test case 1: Euler, M=0.5, $\alpha = 2^\circ$

Coarse grid:

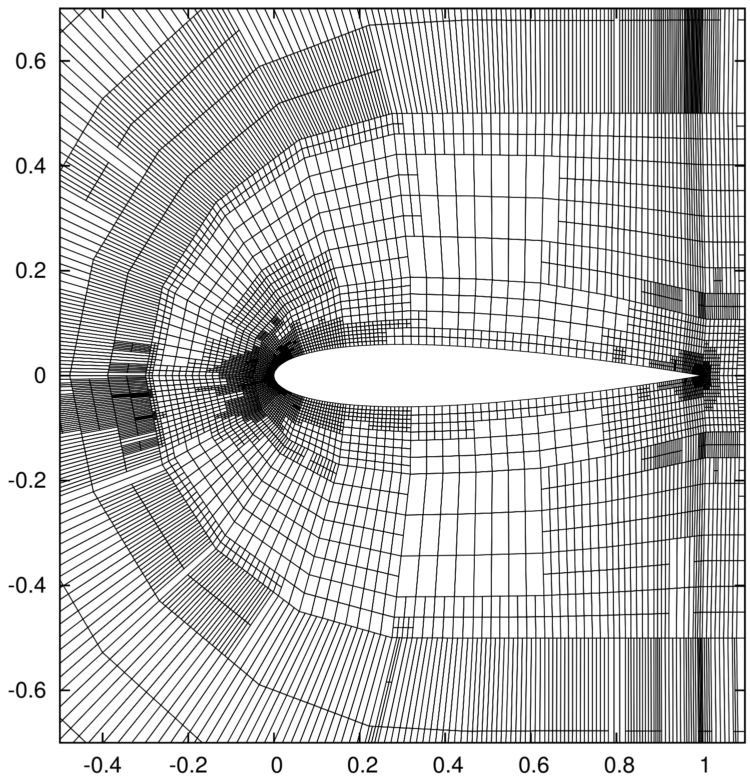




Bad aspect ratios will be inherited to isotropically refined elements

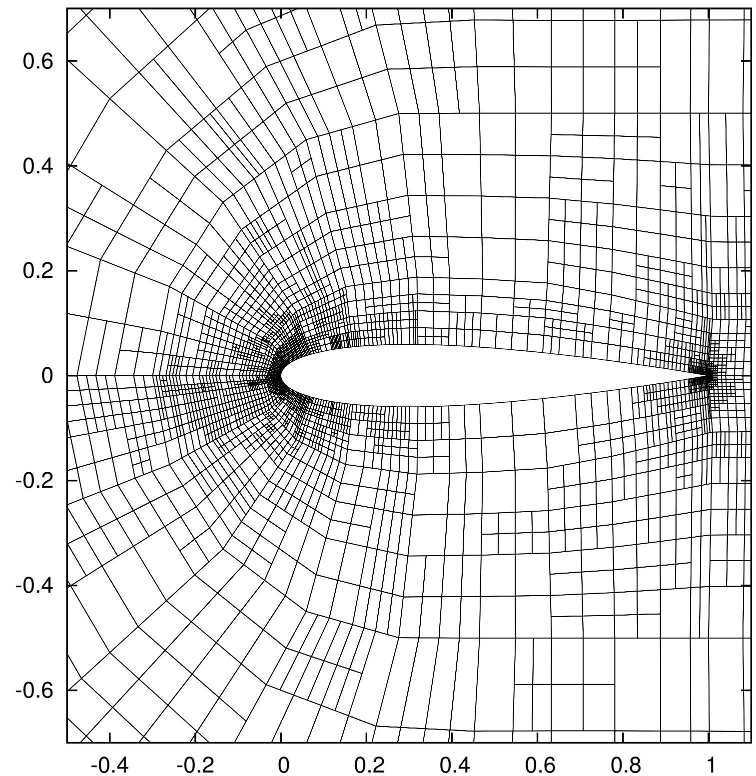
Test case 1: Euler, M=0.5, $\alpha = 2^\circ$

Goal-oriented (adjoint-based) refinement: target quantity $J(u) = c_{lp}$



isotropic (16282 elements)

error in c_{lp} : $1.4\,10^{-4}$



anisotropic (5403 elements)

error in c_{lp} : $1.1\,10^{-4}$

Outlook: Next steps towards application in industry

- ▶ Extension of the flow solver
  - to three-dimensional turbulent, high Reynolds flow
- ▶ The above extension also for
  - the computation of the adjoint solution
  - the evaluation of refinement indicators (residual-based and adjoint-based)
  - and the error estimation with respect to aerodynamical force coefficients
- ▶ Error estimation and adaptivity with respect to multiple target quantities
- ▶ Extension to hybrid meshes
- ▶ Extension to hp-refinement
- ▶ Efficient solution algorithms
  - linear and nonlinear multigrid
  - h- and p- multigrid

EU Project: ADIGMA

Adaptive Higher-Order Variational Methods for Aerodynamic Applications in Industry

Start was 1st of Sept 2006

Co-ordinator:	DLR
Industrial partners:	Airbus-D, Airbus-F, Dassault, Alenia, EADS-M
Research institutes:	DLR, ONERA, NLR, FOI, INRIA, VKI
Universities:	Uni Bergamo, Uni Twente, Uni Swansea, Uni Nottingham
	Uni Stuttgart, Uni Warsaw, Uni Prague, ENSAM, (Uni Nanjing)
SMEs:	ARA, CENAERO

Topics: Mainly Discontinuous Galerkin methods, also Residual Distribution Schemes
Multigrid, Newton-like methods, adaptation, error estimation, hp-refinement